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Five-dimensional $\mathcal{N} = 1$ AdS superspace: Geometry, off-shell multiplets and dynamics

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Abstract

As a step towards formulating projective superspace techniques for supergravity theories with eight supercharges, this work is devoted to field theory in five-dimensional $\mathcal{N} = 1$ anti-de Sitter superspace $\mathrm{AdS}^{5|8} = \mathrm{SU}(2,2|1)/\mathrm{SO}(4,1) \times \mathrm{U}(1)$ which is a maximally symmetric curved background. We develop the differential geometry of $\mathrm{AdS}^{5|8}$ and describe its isometries in terms of Killing supervectors. Various off-shell supermultiplets in $\mathrm{AdS}^{5|8} \times S^2$ are defined, and supersymmetric actions are constructed both in harmonic and projective superspace approaches. Several families of supersymmetric theories are presented including nonlinear sigma-models, Chern-Simons theories and vector-tensor dynamical systems. Using a suitable coset representative, we make use of the coset construction to develop an explicit realization for one half of the superspace $\mathrm{AdS}^{5|8}$ as a trivial fiber bundle with fibers isomorophic to four-dimensional Minkowski superspace.

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Contents

1 Introduction

In four-dimensional $\mathcal{N} = 2$ Poincaré supersymmetry, there exist two powerful formalisms to construct off-shell manifestly supersymmetic actions: harmonic superspace [1, 2] and projective superspace [3, 4, 5, 6]. Both approaches make use of the superspace $\mathbb{R}^{4|8} \times S^2$ and its supersymmetric subspaces, which were introduced for the first time by Rosly [7] who built on earlier ideas due to Witten [8]. Both approaches can naturally be extended to the case of d-dimensional supersymmetry with eight supercharges, for $d \leq 6$, where the appropriate flat superspace with auxiliary bosonic dimensions is $\mathbb{R}^{d|8} \times S^2$. Specifically, the harmonic superspace formulations were developed in [9, 10] for $d = 5$, and in [11] for $d = 6$. The projective superspace formulations were developed in [10] for $d = 5$, and in [12, 13] for $d = 6$.

In projective superspace, off-shell multiplets are reasonably short and can readily be expressed in terms of 4D $\mathcal{N} = 1$ superfields. The latter property is very appealing from the point of view of brane(-world) models. It is also expected [14, 15] that projective superspace should be relevant in the context of hybrid string theory [16]. For these and similar possible applications, one actually needs projective superspace techniques for supergravity. So far, to the best of our knowledge, the projective superspace approach has been mastered only in the flat case.

In harmonic superspace, the prepotential structure of 4D $\mathcal{N} = 2$ supergravity is well understood [17, 2], and similar constructions are clearly applicable in five and six dimensions, see [18] for the six-dimensional case. What is still missing here, in our opinion, is a properly incorporated covariant formalism of differential geometry for superfield supergravity, which should be similar in spirit to the famous Wess-Zumino approach to (the old minimal formulation for) 4D $\mathcal{N} = 1$ supergravity reviewed in [19]. In four-dimensional $\mathcal{N} = 1$ supergravity, it has been recognized for a long time that the most efficient approach to superfield supergravity occurs if one merges together and uses, depending on a concrete application, both the covariant and prepotential techniques [20, 21].

Unlike the purely prepotential approach pursued in [17, 2], this paper is targeted at (making the first step towards) developing covariant superfield techniques for supergravity theories with eight supercharges. Our point of departure is as follows. It is known that all information about off-shell supergravity formulations (including the structure of possible matter multiplets) is encoded in the corresponding algebra of covariant derivatives. We would like to use only this input and try to develop techniques to construct supersymmetric actions both in the harmonic and projective settings. In this paper we consider one

particular supergravity background – five-dimensional $\mathcal{N}=1$ anti-de Sitter superspace, $\mathrm{AdS}^{5|8}$, and explicitly develop harmonic and projective formulations in a covariant fashion using only the language of differential geometry. We believe that similar ideas should be applicable for a general supergravity background, as well as in four and six space-time dimensions. In particular, the case of 4D $\mathcal{N}=2$ anti-de Sitter superspace[1] can be treated similarly.

This paper is organized as follows. In section 2 we derive the algebra of the covariant derivatives for 5D $\mathcal{N}=1$ anti-de Sitter superspace by solving the Bianchi identities. In section 3 the isometries of $\mathrm{AdS}^{5|8}$ are realized in terms of Killing supervectors. In section 4 we introduce analytic multiplets over the harmonic superspace $\mathrm{AdS}^{5|8}\times S^2$ and formulate the harmonic superspace action. Various projective multiplets are defined in section 5, as well as the projective superspace action is formulated. A remarkable feature of this supersymmetric action is that it is uniquely determined by two independent requirements: (i) projective invariance; (ii) invariance under the isometry group $\mathrm{SU}(2,2|1)$. Some important examples of dynamical systems in the AdS projective superspace are given in section 6. An explicit coset construction for one half of $\mathrm{AdS}^{5|8}$ (Poincaré chart) is elaborated in section 7. Our 5D notation and conventions are collected in Appendix A.

2 Covariant derivatives

In this section, we develop the differential geometry of five-dimensional $\mathcal{N}=1$ anti-de Sitter superspace, $\mathrm{AdS}^{5|8}$. This is a supersymmetric version of spaces of constant curvature and, similar to all symmetric spaces, it can be realized as a coset space, specifically $\mathrm{AdS}^{5|8}=\mathrm{SU}(2,2|1)/\mathrm{SO}(4,1)\times\mathrm{U}(1)$. Group-theoretical aspects of $\mathrm{AdS}^{5|8}$ will be discussed in section 7.

Let $z^{\hat{M}}=(x^{\hat{m}},\theta^{\hat{\mu}}_i)$ be local bosonic (x) and fermionic (θ) coordinates parametrizing $\mathrm{AdS}^{5|8}$, where $\hat{m}=0,1,\cdots,4$, $\hat{\mu}=1,\cdots,4$, and $i=\underline{1},\underline{2}$. The Grassmann variables $\theta^{\hat{\mu}}_i$ are assumed to obey a standard pseudo-Majorana reality condition. Since the holonomy group of $\mathrm{AdS}^{5|8}$ is $\mathrm{SO}(4,1)\times\mathrm{U}(1)$, the superspace covariant derivative $\mathcal{D}_{\hat{A}}=(\mathcal{D}_{\hat{a}},\mathcal{D}^i_{\hat{\alpha}})$ can be chosen to have the form

$$\mathcal{D}_{\hat{A}} \;=\; E_{\hat{A}} \;+\; \mathrm{i}\,\Phi_{\hat{A}}\,J \;+\; \frac{1}{2}\,\Omega_{\hat{A}}{}^{\hat{b}\hat{c}}\,M_{\hat{b}\hat{c}} \;=\; E_{\hat{A}} \;+\; \mathrm{i}\,\Phi_{\hat{A}}\,J \;+\; \Omega_{\hat{A}}{}^{\hat{\beta}\hat{\gamma}}\,M_{\hat{\beta}\hat{\gamma}}\;. \qquad (2.1)$$

[1]The 4D $\mathcal{N}=2$ anti-de Sitter superspace was studied in detail in [22] where a manifestly supersymmetric formulation for the off-shell 4D $\mathcal{N}=2$ anti-de Sitter higher spin supermultiplets [23] was given. A few years later, some formal aspects of this superspace were also discussed in [24].

Here $E_{\hat{A}} = E_{\hat{A}}{}^{\hat{M}}(z)\partial_{\hat{M}}$ is the supervielbein, with $\partial_{\hat{M}} = \partial/\partial z^{\hat{M}}$, J the Hermitian generator of the group U(1), $M_{\hat{b}\hat{c}}$ the generators of the Lorentz group SO(4, 1), and $\Phi_{\hat{A}}(z)$ and $\Omega_{\hat{A}}{}^{\hat{b}\hat{c}}(z)$ the corresponding connections. The Lorentz generators with vector indices ($M_{\hat{a}\hat{b}} = -M_{\hat{b}\hat{a}}$) and spinor indices ($M_{\hat{\alpha}\hat{\beta}} = M_{\hat{\beta}\hat{\alpha}}$) are related to each other by the rule: $M_{\hat{a}\hat{b}} = (\Sigma_{\hat{a}\hat{b}})^{\hat{\alpha}\hat{\beta}} M_{\hat{\alpha}\hat{\beta}}$, see Appendix A for more details regarding our 5D notation and conventions. The generators of the holonomy group act on the covariant derivatives as follows:

$$[J, \mathcal{D}^i_{\hat{\alpha}}] = J^i{}_j \mathcal{D}^j_{\hat{\alpha}} \ , \tag{2.2}$$

$$[M_{\hat{\alpha}\hat{\beta}}, \mathcal{D}^i_{\hat{\gamma}}] = \frac{1}{2}\Big(\varepsilon_{\hat{\gamma}\hat{\alpha}}\mathcal{D}^i_{\hat{\beta}} + \varepsilon_{\hat{\gamma}\hat{\beta}}\mathcal{D}^i_{\hat{\alpha}}\Big) \ . \tag{2.3}$$

The Hermitian matrix $J^i{}_j$ should be traceless, $J^i{}_i = 0$, in order to preserve the pseudo-Majorana condition enjoyed by the covariant derivatives. The latter condition is equivalent to the fact that the isotensors[2] $J^{ij} = \varepsilon^{jk} J^i{}_k$ and $J_{ij} = \varepsilon_{ik} J^k{}_j$ are symmetric, $J^{ij} = J^{ji}$, $J_{ij} = J_{ji}$. The fact that $J^i{}_j$ is Hermitian, can be seen to be equivalent to $(J^{ij})^* = -J_{ij}$.

The algebra of covariant derivatives can be reconstructed if we impose the following two requirements: (i) the torsion tensor is covariantly constant;[3] (ii) the group SO(4, 1) × U(1) belongs to the automorphism group. These requirements lead, in particular, to the ansätze:

$$\{\mathcal{D}^i_{\hat{\alpha}}, \mathcal{D}^j_{\hat{\beta}}\} = -2\mathrm{i}\varepsilon^{ij}\mathcal{D}_{\hat{\alpha}\hat{\beta}} + x\varepsilon^{ij}\varepsilon_{\hat{\alpha}\hat{\beta}} J + f^{ij} M_{\hat{\alpha}\hat{\beta}} \ , \tag{2.4}$$

$$[\mathcal{D}_{\hat{a}}, \mathcal{D}^i_{\hat{\alpha}}] = C^i{}_j (\Gamma_{\hat{a}})_{\hat{\alpha}}{}^{\hat{\beta}} \mathcal{D}^j_{\hat{\beta}} \ , \tag{2.5}$$

where

$$\mathcal{D}_{\hat{\alpha}\hat{\beta}} = (\Gamma^{\hat{a}})_{\hat{\alpha}\hat{\beta}} \mathcal{D}_{\hat{a}} \ , \tag{2.6}$$

and x is a constant parameters, $f^{ij} = f^{ji}$, $C^i{}_j$ is a 2×2 matrix. Eq. (2.5) can be rewritten in the equivalent form

$$[\mathcal{D}_{\hat{\alpha}\hat{\beta}}, \mathcal{D}^i_{\hat{\gamma}}] = -2C^i{}_j\Big(\varepsilon_{\hat{\gamma}\hat{\alpha}}\mathcal{D}^j_{\hat{\beta}} - \varepsilon_{\hat{\gamma}\hat{\beta}}\mathcal{D}^j_{\hat{\alpha}} + \frac{1}{2}\varepsilon_{\hat{\alpha}\hat{\beta}}\mathcal{D}^j_{\hat{\gamma}}\Big) \ , \tag{2.7}$$

Note that setting $x = m = C^i{}_j = 0$ gives the flat supersymmetry algebra, see. e.g., [10].

The (covariantly) constant parameters x, f^{ij} and $C^i{}_j$ in (2.4) and (2.5) turn out to be considerably constrained on general grounds. Firstly, the tensor f^{ij} must be invariant under the action of J,

$$J f^{ij} = (J^i{}_k f^{kj} + J^j{}_k f^{ik}) = 0 \quad \Longleftrightarrow \quad f^{ij} = mJ^{ij} \ , \tag{2.8}$$

[2]Two-component indices i, j are raised and lowered using the SL(2, $\mathbb{C}$)-invariant antisymmetric tensors ε^{ij} and ε_{ij} normalized by $\varepsilon^{ik}\varepsilon_{kj} = \delta^i_j$ and $\varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1$.

[3]Then, in accordance with Dragon's theorem [26], the curvature tensor is covariantly constant.

with m a constant parameter. Secondly, we should take care of reality conditions such as

$$(\mathcal{D}^i_{\hat\alpha} F)^* = -(-1)^{\epsilon(F)} \mathcal{D}^{\hat\alpha}_i F^* , \tag{2.9}$$

where $\epsilon(F)$ is the Grassmann parity of F. They imply that

$$x \;=\; \overline{x} , \qquad m \;=\; \overline{m} . \tag{2.10}$$

and $C^i{}_j$ is anti-Hermitian,

$$C^\dagger = -C , \qquad C = (C^i{}_j) . \tag{2.11}$$

Of course, we should also guarantee the fulfillment of the Bianchi identities, and this proves to lead to additional restrictions on the parameters. In particular, the dimension-3/2 Bianchi identity

$$[\mathcal{D}^i_{\hat\alpha}, \{\mathcal{D}^j_{\hat\beta}, \mathcal{D}^k_{\hat\gamma}\}] + [\mathcal{D}^j_{\hat\beta}, \{\mathcal{D}^k_{\hat\gamma}, \mathcal{D}^i_{\hat\alpha}\}] + [\mathcal{D}^k_{\hat\gamma}, \{\mathcal{D}^i_{\hat\alpha}, \mathcal{D}^j_{\hat\beta}\}] \;=\; 0 \tag{2.12}$$

can be shown to imply

$$C^i{}_j = \frac{\mathrm{i}}{2} \omega J^i{}_j , \qquad \omega = \Big(\frac{m}{2} - x\Big) . \tag{2.13}$$

Imposing the dimension-2 Bianchi identity

$$[\mathcal{D}_{\hat a}, \{\mathcal{D}^i_{\hat\alpha}, \mathcal{D}^j_{\hat\beta}\}] + \{\mathcal{D}^i_{\hat\alpha}, [\mathcal{D}^j_{\hat\beta}, \mathcal{D}_{\hat a}]\} - \{\mathcal{D}^j_{\hat\beta}, [\mathcal{D}_{\hat a}, \mathcal{D}^i_{\hat\alpha}]\} \;=\; 0 \tag{2.14}$$

leads, in particular, to

$$\begin{aligned} [\mathcal{D}_{\hat a}, \mathcal{D}_{\hat b}] &= -\frac{\mathrm{i}}{16} \varepsilon_{ij} (\Gamma_{\hat b})^{\hat\alpha\hat\beta} [\mathcal{D}_{\hat a}, \{\mathcal{D}^i_{\hat\alpha}, \mathcal{D}^j_{\hat\beta}\}] \\ &= \frac{\mathrm{i}}{16} \varepsilon_{ij} (\Gamma_{\hat b})^{\hat\alpha\hat\beta} \Big(\{\mathcal{D}^i_{\hat\alpha}, [\mathcal{D}^j_{\hat\beta}, \mathcal{D}_{\hat a}]\} - \{\mathcal{D}^j_{\hat\beta}, [\mathcal{D}_{\hat a}, \mathcal{D}^i_{\hat\alpha}]\} \Big) , \end{aligned} \tag{2.15}$$

and then

$$[\mathcal{D}_{\hat a}, \mathcal{D}_{\hat b}] = \frac{1}{4} m \omega J^2 M_{\hat a\hat b} , \tag{2.16}$$

where

$$J^2 \;\equiv\; -\frac{1}{2} J^{ij} J_{ij} . \tag{2.17}$$

Another consequence of the dimension-2 Bianchi identity (2.14) is

$$\omega \;=\; -\frac{1}{4} m . \tag{2.18}$$

As a result, all the parameters in (2.4) and (2.5) have been expressed in terms of ω. With the above conditions taken into account, the remaining dimension-$\frac{5}{2}$ Bianchi identity

$$[\mathcal{D}_{\hat{a}}, [\mathcal{D}_{\hat{b}}, \mathcal{D}^i_{\hat{\alpha}}]] + [\mathcal{D}_{\hat{b}}, [\mathcal{D}^i_{\hat{\alpha}}, \mathcal{D}_{\hat{a}}]] + [\mathcal{D}^i_{\hat{\alpha}}, [\mathcal{D}_{\hat{a}}, \mathcal{D}_{\hat{b}}]] \;=\; 0 \;, \tag{2.19}$$

and dimension-3 Bianchi identity

$$[\mathcal{D}_{\hat{a}}, [\mathcal{D}_{\hat{b}}, \mathcal{D}_{\hat{c}}]] + [\mathcal{D}_{\hat{b}}, [\mathcal{D}_{\hat{c}}, \mathcal{D}_{\hat{a}}]] + [\mathcal{D}_{\hat{c}}, [\mathcal{D}_{\hat{a}}, \mathcal{D}_{\hat{b}}]] \;=\; 0 \;, \tag{2.20}$$

are satisfied identically.

Let us summarise the results obtained. The covariant derivatives for $\mathrm{AdS}^{5|8}$ obey the algebra

$$\{\mathcal{D}^i_{\hat{\alpha}}, \mathcal{D}^j_{\hat{\beta}}\} = -2\mathrm{i}\varepsilon^{ij}\mathcal{D}_{\hat{\alpha}\hat{\beta}} - 3\omega\varepsilon^{ij}\varepsilon_{\hat{\alpha}\hat{\beta}}J - 4\omega J^{ij}M_{\hat{\alpha}\hat{\beta}} \;, \tag{2.21a}$$

$$[\mathcal{D}_{\hat{a}}, \mathcal{D}^i_{\hat{\beta}}] = \frac{\mathrm{i}}{2}\omega J^i{}_j(\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\gamma}}\mathcal{D}^j_{\hat{\gamma}} \;, \tag{2.21b}$$

$$[\mathcal{D}_{\hat{a}}, \mathcal{D}_{\hat{b}}] = -\omega^2 J^2 M_{\hat{a}\hat{b}} \;. \tag{2.21c}$$

It is useful to rewrite (2.21b) in the equivalent form

$$[\mathcal{D}_{\hat{\alpha}\hat{\beta}}, \mathcal{D}^i_{\hat{\gamma}}] = -\mathrm{i}\,\omega J^i{}_j\left(\varepsilon_{\hat{\gamma}\hat{\alpha}}\mathcal{D}^j_{\hat{\beta}} - \varepsilon_{\hat{\gamma}\hat{\beta}}\mathcal{D}^j_{\hat{\alpha}} + \frac{1}{2}\varepsilon_{\hat{\alpha}\hat{\beta}}\mathcal{D}^j_{\hat{\gamma}}\right) \;. \tag{2.22}$$

As follows from (2.21c), the bosonic body of the superspace is characterised by a constant negative curvature, and therefore it is AdS_5. Indeed, since $J^i{}_j$ is Hermitian and traceless, we have

$$J^i{}_j \;=\; J^I(\sigma_I)^i{}_j \;\Longrightarrow\; J^2 = -\frac{1}{2}J^{ij}J_{ij} = \frac{1}{2}J^IJ^J\,\mathrm{tr}(\sigma_I\sigma_J) = J^IJ_I \;>\; 0 \;, \tag{2.23}$$

where J^I is a real tree-vector, with $I = 1, 2, 3$, and σ^I are the Pauli matrices. In section 7, we will give an explicit (coset space) realization of the geometry described.

Up to an isomorphism, one can always choose $J^i{}_j \propto (\sigma_3)^i{}_j$, and hence $J^{\underline{11}} = J^{\underline{22}} = 0$. Then, it follows from (2.21a–2.21c) that each of the two subsets of covariant derivatives $(\mathcal{D}_{\hat{a}}, \mathcal{D}^{\underline{1}}_{\hat{\alpha}})$ and $(\mathcal{D}_{\hat{a}}, \mathcal{D}^{\underline{2}}_{\hat{\alpha}})$ forms a closed algebra, in particular

$$\{\mathcal{D}^{\underline{1}}_{\hat{\alpha}}, \mathcal{D}^{\underline{1}}_{\hat{\beta}}\} = 0 \;, \tag{2.24a}$$

$$[\mathcal{D}_{\hat{a}}, \mathcal{D}^{\underline{1}}_{\hat{\beta}}] = \frac{\mathrm{i}}{2}\omega J^{\underline{1}}{}_{\underline{1}}(\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\gamma}}\mathcal{D}^{\underline{1}}_{\hat{\gamma}} \;, \tag{2.24b}$$

$$[\mathcal{D}_{\hat{a}}, \mathcal{D}_{\hat{b}}] = -\omega^2 J^2 M_{\hat{a}\hat{b}} \;. \tag{2.24c}$$

Therefore, one can consistently define covariantly chiral superfields obeying the constraint $\mathcal{D}^2_{\hat\alpha}\Phi = 0$. Unlike the case of 4D $\mathcal{N} = 1$ anti-de Sitter superspace [27], such multiplets can transform in arbitrary representations of the Lorentz group.

In what follows, it will be useful to deal with a different basis for the spinor covariant derivatives. Let us introduce two linearly independent isospinors u_i^+ and u_i^-,

$$u^{+i}u_i^- \equiv (u^+u^-) \neq 0 \quad \Longrightarrow \quad \delta_j^i = \frac{1}{(u^+u^-)}(u^{+i}u_j^- - u^{-i}u_j^+) , \tag{2.25}$$

which do not transform under the action of J, that is $J\,u_i^+ = J\,u_i^- = 0$. Then, defining

$$\mathcal{D}^\pm_{\hat\alpha} \equiv \mathcal{D}^i_{\hat\alpha}u_i^\pm , \tag{2.26}$$

$$J^{++} \equiv J^{ij}u_i^+u_j^+ , \qquad J^{+-} \equiv J^{ij}u_i^+u_j^- , \qquad J^{--} \equiv J^{ij}u_i^-u_j^- . \tag{2.27}$$

the relations (2.21a) and (2.21b) become

$$\{\mathcal{D}^+_{\hat\alpha}, \mathcal{D}^+_{\hat\beta}\} = -4\omega J^{++}M_{\hat\alpha\hat\beta} , \tag{2.28a}$$

$$\{\mathcal{D}^+_{\hat\alpha}, \mathcal{D}^-_{\hat\beta}\} = 2(u^+u^-)\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta} + 3(u^+u^-)\omega\varepsilon_{\hat\alpha\hat\beta}J - 4\omega J^{+-}M_{\hat\alpha\hat\beta} , \tag{2.28b}$$

$$\{\mathcal{D}^-_{\hat\alpha}, \mathcal{D}^-_{\hat\beta}\} = -4\omega J^{--}M_{\hat\alpha\hat\beta} , \tag{2.28c}$$

$$[\mathcal{D}_{\hat a}, \mathcal{D}^+_{\hat\alpha}] = -\frac{\mathrm{i}\omega}{2(u^+u^-)}(\Gamma_{\hat a})_{\hat\alpha}{}^{\hat\beta}(J^{++}\mathcal{D}^-_{\hat\beta} - J^{+-}\mathcal{D}^+_{\hat\beta}) , \tag{2.28d}$$

$$[\mathcal{D}_{\hat a}, \mathcal{D}^-_{\hat\alpha}] = \frac{\mathrm{i}\omega}{2(u^+u^-)}(\Gamma_{\hat a})_{\hat\alpha}{}^{\hat\beta}(J^{--}\mathcal{D}^+_{\hat\beta} - J^{+-}\mathcal{D}^-_{\hat\beta}) . \tag{2.28e}$$

Eqs. (2.28d) and (2.28e) are equivalent to

$$\begin{aligned}[\mathcal{D}_{\hat\alpha\hat\beta}, \mathcal{D}^+_{\hat\gamma}] = {}& \frac{\mathrm{i}\omega}{(u^+u^-)}J^{++}\left(\varepsilon_{\hat\gamma\hat\alpha}\mathcal{D}^-_{\hat\beta} - \varepsilon_{\hat\gamma\hat\beta}\mathcal{D}^-_{\hat\alpha} + \frac{1}{2}\varepsilon_{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\gamma}\right) \\ & -\frac{\mathrm{i}\omega}{(u^+u^-)}J^{+-}\left(\varepsilon_{\hat\gamma\hat\alpha}\mathcal{D}^+_{\hat\beta} - \varepsilon_{\hat\gamma\hat\beta}\mathcal{D}^+_{\hat\alpha} + \frac{1}{2}\varepsilon_{\hat\alpha\hat\beta}\mathcal{D}^+_{\hat\gamma}\right) ,\end{aligned} \tag{2.29a}$$

$$\begin{aligned}[\mathcal{D}_{\hat\alpha\hat\beta}, \mathcal{D}^-_{\hat\gamma}] = {}& \frac{\mathrm{i}\omega}{(u^+u^-)}J^{+-}\left(\varepsilon_{\hat\gamma\hat\alpha}\mathcal{D}^-_{\hat\beta} - \varepsilon_{\hat\gamma\hat\beta}\mathcal{D}^-_{\hat\alpha} + \frac{1}{2}\varepsilon_{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\gamma}\right) \\ & -\frac{\mathrm{i}\omega}{(u^+u^-)}J^{--}\left(\varepsilon_{\hat\gamma\hat\alpha}\mathcal{D}^+_{\hat\beta} - \varepsilon_{\hat\gamma\hat\beta}\mathcal{D}^+_{\hat\alpha} + \frac{1}{2}\varepsilon_{\hat\alpha\hat\beta}\mathcal{D}^+_{\hat\gamma}\right) .\end{aligned} \tag{2.29b}$$

Under general coordinate and local SO(4,1)×U(1) transfomations, the covariant derivatives change as

$$\mathcal{D}_{\hat A} \to \mathcal{D}'_{\hat A} = \mathrm{e}^{\tau}\,\mathcal{D}_{\hat A}\,\mathrm{e}^{-\tau} , \qquad \tau = \tau^{\hat B}(z)\mathcal{D}_{\hat B} + \mathrm{i}\,\tau(z)J + \tau^{\hat\beta\hat\gamma}(z)M_{\hat\beta\hat\gamma} . \tag{2.30}$$

This gauge freedom can be used to impose a suitable Wess-Zumino gauge. The latter can be chosen such that

$$\mathcal{D}_{\hat{a}}| = \nabla_{\hat{a}} = e_{\hat{a}}{}^{\hat{m}}(x)\,\partial_{\hat{m}} + \frac{1}{2}\omega_{\hat{a}}{}^{\hat{b}\hat{c}}(x)\,M_{\hat{b}\hat{c}}~, \tag{2.31}$$

where $U|$ means the θ independent part of a superfield $U(x,\theta)$,

$$U = U(z) = U(x,\theta)~, \qquad U| = U(x,\theta=0)~. \tag{2.32}$$

and $\nabla_{\hat{a}}$ stands for the covariant derivatives of anti-de Sitter space,

$$[\nabla_{\hat{a}},\nabla_{\hat{b}}] = -\omega^2 J^2 M_{\hat{a}\hat{b}}~. \tag{2.33}$$

3 Killing supervectors

Similar to the 4D $\mathcal{N}=1$ case [21], the isometry group $\mathrm{SU}(2,2|1)$ of $\mathrm{AdS}^{5|8}$ is generated by those supervector fields $\xi^{\hat{A}}(z)E_{\hat{A}}$ which enjoy the property

$$\delta_\xi \mathcal{D}_{\hat{A}} = -[(\xi + \mathrm{i}\rho J + \Lambda^{\hat{\beta}\hat{\gamma}}M_{\hat{\beta}\hat{\gamma}}),\mathcal{D}_{\hat{A}}] = 0~, \tag{3.1}$$

where

$$\xi \equiv \xi^{\hat{A}}\mathcal{D}_{\hat{A}} = \xi^{\hat{a}}\mathcal{D}_{\hat{a}} + \xi^{\hat{\alpha}}_i\mathcal{D}^i_{\hat{\alpha}} = -\frac{1}{4}\xi^{\hat{\alpha}\hat{\beta}}\mathcal{D}_{\hat{\alpha}\hat{\beta}} + \xi^{\hat{\alpha}}_i\mathcal{D}^i_{\hat{\alpha}}~, \tag{3.2}$$

for some real scalar $\rho(z)$ and symmetric tensor $\Lambda^{\hat{\beta}\hat{\gamma}}(z) = \Lambda^{\hat{\gamma}\hat{\beta}}(z)$. The $\xi^{\hat{A}}(z)E_{\hat{A}}$ is called a Killing supervector. The set of all Killing supervectors forms a Lie algebra with respect to the Lie bracket. Given a Killing supervector, it generates a symmetry transformation of matter superfields, which live on $\mathrm{AdS}^{5|8}$, defined as

$$\delta_\xi\chi = -(\xi + \mathrm{i}\,\rho J + \Lambda^{\hat{\alpha}\hat{\beta}}M_{\hat{\alpha}\hat{\beta}})\chi~. \tag{3.3}$$

Using the (anti) commutation relations (2.21a) – (2.21c), eq. (3.1) can be seen to be equivalent to

$$\begin{aligned} 0 \;=\; & \Big(\frac{1}{4}\mathcal{D}^i_{\hat{\alpha}}\xi^{\hat{\beta}\hat{\gamma}} + 2\mathrm{i}\xi^{i\hat{\beta}}\delta^{\hat{\gamma}}_{\hat{\alpha}}\Big)\mathcal{D}_{\hat{\beta}\hat{\gamma}} + \Big(\frac{\mathrm{i}}{2}\omega\,\xi_{\hat{\alpha}}{}^{\hat{\beta}}J^i{}_j - \mathcal{D}^i_{\hat{\alpha}}\xi^{\hat{\beta}}_j + \mathrm{i}\,\rho J^i{}_j\delta^{\hat{\beta}}_{\hat{\alpha}} + \delta^i_j\Lambda_{\hat{\alpha}}{}^{\hat{\beta}}\Big)\mathcal{D}^j_{\hat{\beta}} \\ & -\Big(3\omega\,\xi^i_{\hat{\alpha}} + \mathrm{i}\mathcal{D}^i_{\hat{\alpha}}\rho\Big)J - \Big(2\omega J^{ij}(\xi^{\hat{\beta}}_j\delta^{\hat{\gamma}}_{\hat{\alpha}} + \xi^{\hat{\gamma}}_j\delta^{\hat{\beta}}_{\hat{\alpha}}) + \mathcal{D}^i_{\hat{\alpha}}\Lambda^{\hat{\beta}\hat{\gamma}}\Big)M_{\hat{\beta}\hat{\gamma}}~, \end{aligned} \tag{3.4}$$

and from here we deduce the set of Killing supervector equations

$$\mathcal{D}^i_{\hat{\alpha}}\xi^{\hat{a}} = -2\mathrm{i}(\Gamma^{\hat{a}})_{\hat{\alpha}\hat{\beta}}\,\xi^{i\hat{\beta}}~, \tag{3.5a}$$

$$0 = \frac{\mathrm{i}}{2}\omega\,\xi_{\hat{\alpha}\hat{\beta}}J^i{}_j - \mathcal{D}^i_{\hat{\alpha}}\xi_{j\hat{\beta}} - \mathrm{i}\,\rho J^i{}_j\varepsilon_{\hat{\alpha}\hat{\beta}} + \delta^i_j\Lambda_{\hat{\alpha}\hat{\beta}}~, \tag{3.5b}$$

$$\mathrm{i}\,\mathcal{D}^i_{\hat{\alpha}}\rho = -3\omega\,\xi^i_{\hat{\alpha}}~, \tag{3.5c}$$

$$\mathcal{D}^i_{\hat{\alpha}}\Lambda^{\hat{\beta}\hat{\gamma}} = -2\omega J^{ij}(\xi^{\hat{\beta}}_j\delta^{\hat{\gamma}}_{\hat{\alpha}} + \xi^{\hat{\gamma}}_j\delta^{\hat{\beta}}_{\hat{\alpha}})~. \tag{3.5d}$$

Note that (3.5b) is equivalent to the following equations

$$\mathcal{D}^i_{\hat\alpha}\xi_{i\hat\beta} = 2\Lambda_{\hat\alpha\hat\beta} ~, \tag{3.6a}$$

$$\mathcal{D}^{i\hat\alpha}\xi^j_{\hat\alpha} + \mathcal{D}^{j\hat\alpha}\xi^i_{\hat\alpha} = 8{\rm i}\, J^{ij}\rho ~, \tag{3.6b}$$

$$(\Gamma_{\hat a})^{\hat\alpha\hat\beta}(\mathcal{D}^i_{\hat\alpha}\xi^j_{\hat\beta} + \mathcal{D}^j_{\hat\alpha}\xi^i_{\hat\beta}) = -4{\rm i}\omega\, J^{ij}\xi_{\hat a} ~. \tag{3.6c}$$

It is seen that the parameters of U(1) and Lorentz transformations, ρ and $\Lambda_{\hat\alpha\hat\beta}$, are uniquely expressed in terms of the spinor components of the Killing supervector. As to the vector components $\xi^{\hat a}$ of ξ,which is also uniquely determined in terms of the spinor components of ξ, it obeys the standard Killing equation

$$\mathcal{D}^{(\hat a}\xi^{\hat b)} = 0 ~. \tag{3.7}$$

To prove (3.7), it suffices to represent $\mathcal{D}^{\hat a}$ in (3.7) in the form

$$\mathcal{D}^{\hat a} = \frac{\rm i}{8}(\Gamma^{\hat a})^{\hat\alpha\hat\beta}\mathcal{D}^i_{\hat\alpha}\mathcal{D}_{i\hat\beta} ~, \tag{3.8}$$

and then make use of relations (3.5a) and (3.6a).

As is seen from eqs. (3.5c), (3.6a) and (3.6c), the components of ξ (hence, the Lorentz parameter $\Lambda_{\hat\alpha\hat\beta}$ as well) can be expressed in terms of the scalar parameter ρ as follows:

$$\xi^i_{\hat\alpha} = -\frac{\rm i}{3\omega}\,\mathcal{D}^i_{\hat\alpha}\rho ~, \tag{3.9a}$$

$$\xi_{\hat a} = -\frac{\rm i}{\omega J^2}J_{ij}(\Gamma_{\hat a})^{\hat\alpha\hat\beta}\,\mathcal{D}^i_{\hat\alpha}\xi^j_{\hat\beta} = -\frac{1}{3\omega^2 J^2}J_{ij}(\Gamma_{\hat a})^{\hat\alpha\hat\beta}\,\mathcal{D}^i_{\hat\alpha}\mathcal{D}^j_{\hat\beta}\,\rho \tag{3.9b}$$

$$\Lambda_{\hat\alpha\hat\beta} = \frac{1}{2}\,\mathcal{D}^i_{\hat\alpha}\xi_{i\hat\beta} = -\frac{\rm i}{6\omega}\,\mathcal{D}^i_{\hat\alpha}\mathcal{D}_{i\hat\beta}\,\rho ~. \tag{3.9c}$$

This is similar to the situation in 4D $\mathcal{N}=2$ AdS supersymmetry [23].

We should point out that equation (3.9c) implies

$$0 = \mathcal{D}^{i\hat\alpha}\xi_{i\hat\alpha} = \mathcal{D}^{i\hat\alpha}\mathcal{D}_{i\hat\alpha}\,\rho ~. \tag{3.10}$$

Furthermore, equations (3.5a), (3.5d) and (3.6b) imply

$$0 = \frac{1}{\omega J^2}J_{jk}(\Gamma^{\hat a})^{\hat\beta\hat\gamma}\mathcal{D}^i_{\hat\alpha}\mathcal{D}^j_{\hat\beta}\xi^k_{\hat\gamma} - 2(\Gamma^{\hat a})_{\hat\alpha\hat\beta}\xi^{i\hat\beta} ~, \tag{3.11a}$$

$$0 = \frac{1}{2}\mathcal{D}^i_{\hat\alpha}\mathcal{D}^{j\hat\beta}\xi^{\hat\gamma}_j + 2\omega J^{ij}(\xi^{\hat\beta}_j\delta^{\hat\gamma}_{\hat\alpha} + \xi^{\hat\gamma}_j\delta^{\hat\beta}_{\hat\alpha}) ~, \tag{3.11b}$$

$$0 = \frac{1}{3\omega}(\mathcal{D}^{i\hat\alpha}\mathcal{D}^j_{\hat\alpha} + \mathcal{D}^{j\hat\alpha}\mathcal{D}^i_{\hat\alpha})\rho + 8J^{ij}\rho ~. \tag{3.11c}$$

From (3.11b) we also deduce

$$\mathcal{D}^i_{\hat\alpha}\mathcal{D}^{j\hat\beta}\xi^{\hat\gamma}_j - \mathcal{D}^i_{\hat\alpha}\mathcal{D}^{j\hat\gamma}\xi^{\hat\beta}_j \;=\; 0 \quad \Longrightarrow \quad \mathcal{D}^i_{\hat\alpha}\mathcal{D}^{\hat\beta\hat\gamma}\rho \;=\; 0 \;, \tag{3.12}$$

and hence

$$\mathcal{D}_{\hat\beta\hat\gamma}\,\rho \;=\; 0 \;. \tag{3.13}$$

We conclude that ρ is annihilated by the vector covariant derivatives.

For later applications, we also observe that the relation $\mathcal{D}^i_{\hat\alpha}\xi^{\hat\alpha}_i = 0$ and eq. (3.5a) imply

$$\mathcal{D}_{\hat\alpha\hat\beta}\,\xi^{i\hat\beta} \;=\; \frac{5}{2}\,\mathrm{i}\,\omega J^i{}_j\xi^j_{\hat\alpha} \;. \tag{3.14}$$

4 Harmonic superspace approach

In the previous two sections, we have described the differential geometry of five-dimensional $\mathcal{N}=1$ AdS superspace and its isometries. From now on, we turn to constructing off-shell supersymmetric theories in $\mathrm{AdS}^{5|8}$. This section is devoted to developing a harmonic superspace approach. To comply with the conventions generally accepted by the harmonic superspace practitioners [1, 2], the isospinors u^+ and u^- in (2.28a–2.28e) will be chosen to obey the following constraints:

$$(u_i{}^-\,,\,u_i{}^+) \in \mathrm{SU}(2)\;, \qquad (u^{+i})^* = u^-_i\;, \qquad (u^+u^-) = 1\;. \tag{4.1}$$

As a first step, it is natural to introduce analytic supermultiplets living on harmonic superspace.

4.1 Analytic multiplets

We start our analysis with the introduction of $O(n)$ supermultiplets living in $\mathrm{AdS}^{5|8}$. Such a multiplet is described by a completely symmetric superfield $H^{i_1\cdots i_n}(z) = H^{(i_1\cdots i_n)}(z)$ (with the symmetrization involving a factor of $1/n!$) constrained to enjoy the analyticity condition[4]

$$\mathcal{D}_{\hat\alpha}{}^{(i_1}H^{i_2\cdots i_{n+1})}(z) \;=\; 0\;. \tag{4.2}$$

[4]In 4D $\mathcal{N}=2$ supersymmetry, off-shell superfields $H^{(i_1\cdots i_n)}(z)$ obeying the constrains $D_\alpha{}^{(i_1}H^{i_2\cdots i_{n+1})}(z) = \bar D_{\dot\alpha}{}^{(i_1}H^{i_2\cdots i_{n+1})}(z) = 0$ have a long history. In the presence of an intrinsic central charge, the cases $n=1$ and $n=2$ correspond to the Fayet-Sohnius hypermultiplet [28] and the linear multiplet [29] respectively. In the absence of central charge, the case $n=2$ corresponds to the tensor multiplet [30]. The case $n=4$ was discussed in [31]. The multiplets with $n>2$ were introduced in [32], in the projective superspace approach, and then re-discovered in [5]. They were called "$O(n)$ multiplets" in [33]. Their harmonic superspace description was given in [34].

It follows from the algebra of covariant derivatives, that this constraint is consistent provided the superfield is scalar with respect to SO(4,1). If one associates with $H^{i_1\cdots i_n}(z)$ a superfield $H^{(n)}(z,u)$ of harmonic charge n,

$$H^{(n)}(z,u) \;=\; u^+_{i_1}\cdots u^+_{i_n}\, H^{i_1\cdots i_n}(z)\ , \tag{4.3}$$

the analyticity condition (4.2) can be seen to be equivalent to

$$\mathcal{D}^+_{\hat\alpha} H^{(n)}(z,u) \;=\; 0\ , \qquad D^{++} H^{(n)}(z,u) \;=\; 0\ . \tag{4.4}$$

Here D^{++} is one of the harmonic derivatives (D^{++}, D^{--}, D^0),

$$\begin{gathered} D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}\ , \quad D^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}\ , \quad D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}\ , \\ [D^0, D^{\pm\pm}] = \pm 2 D^{\pm\pm}\ , \qquad [D^{++}, D^{--}] = D^0\ , \end{gathered} \tag{4.5}$$

which form a basis in the space of left-invariant vector fields for SU(2).

Without imposing the analyticity condition, eq. (4.2), one can consistently define an isotensor superfield $F^{i_1\cdots i_n}(z) = F^{(i_1\cdots i_n)}(z)$ that transforms under the action of the isometry group as follows:

$$\delta_\xi F^{i_1\cdots i_n} \;=\; -\Big(\xi + {\rm i}\,\rho J\Big)\, F^{i_1\cdots i_n} \;=\; -\xi F^{i_1\cdots i_n} - {\rm i}\, q\, n\, \rho J^{(i_1}{}_k F^{i_2\cdots i_n)k}\ , \tag{4.6}$$

where ξ is the Killing supervector, and q is the J-charge of $F^{i_1\cdots i_n}$. One can associate with $F^{i_1\cdots i_n}(z)$ the harmonic superfield $F^{(n)}(z,u) = u^+_{i_1}\cdots u^+_{i_n} F^{i_1\cdots i_n}(z)$. The latter obeys the algebraic constraint $D^{++}F^{(n)} = 0$, and its isometry transformation is

$$\delta_\xi F^{(n)} \;=\; -\xi F^{(n)} + {\rm i}\, q\rho (J^{++}D^{--}F^{(n)} - nJ^{+-}F^{(n)})\ , \tag{4.7}$$

where it has been used the fact that $u^\pm_i$ are inert under the action of J. It is also worth noting that the Killing supervector can be rewritten as

$$\xi = \xi^{\hat a}\mathcal{D}_{\hat a} - \xi^{+\hat\alpha}\mathcal{D}^-_{\hat\alpha} + \xi^{-\hat\alpha}\mathcal{D}^+_{\hat\alpha}\ . \tag{4.8}$$

It is easy to see that the constraint $D^{++}F^{(n)} = 0$ is preserved under the isometry transformations $D^{++}\delta_\xi F^{(n)} = 0$.

If the superfield $F^{(n)}$ is constrained to be analytic, $\mathcal{D}^+_{\hat\alpha} F^{(n)} = 0$, then the value of its J-charge turns out to be uniquely fixed, and namely $q = 1$. Therefore, the isometry transformation of the $O(n)$ multiplet is

$$\delta_\xi H^{(n)} = -\Big(\xi + {\rm i}\,\rho J\Big)\, H^{(n)} = -\Big(\xi^{\hat a}\mathcal{D}_{\hat a} - \xi^{+\hat\alpha}\mathcal{D}^-_{\hat\alpha} - {\rm i}\,\rho\big(J^{++}D^{--} - nJ^{+-}\big)\Big) H^{(n)}\ . \tag{4.9}$$

It is not difficult to extend the above consideration to include more general multiplets. Within the harmonic superspace approach [2], one has to deal with superfields of the form $Q^{(n)}(z,u)$, with n an integer, such that (i) $Q^{(n)}(z,u)$ is a smooth function over the group manifold SU(2) parametrized by $u = ({u_i}^-, {u_i}^+)$; (ii) under harmonic phase transformations $u^\pm \to \exp(\pm \mathrm{i}\varphi) u^\pm$, the charge of $Q^{(n)}(z,u)$ is equal to n,

$$Q^{(n)}(z, \mathrm{e}^{\mathrm{i}\varphi}u^+, \mathrm{e}^{-\mathrm{i}\varphi}u^-) = \mathrm{e}^{n\mathrm{i}\varphi}\, Q^{(n)}(z, u^+, u^-) \;, \quad \Longleftrightarrow \quad D^0 Q^{(n)}(z,u) = nQ^{(n)}(z,u) \;.$$

Such a superfield can be represented by a convergent Fourier series (for definiteness, we choose $n \geq 0$)

$$Q^{(n)}(z,u) = \sum_{k=0}^{+\infty} Q^{(i_1 \cdots i_{k+n} j_1 \cdots j_k)}(z)\, u^+_{i_1} \cdots u^+_{i_{k+n}} u^-_{j_1} \cdots u^-_{j_k} \;. \tag{4.10}$$

To realise an action of the U(1) generator J on $Q^{(n)}$, we define the component superfields in (4.10) to transform by the law:

$$J Q_q^{(i_1 \cdots i_{k+n} j_1 \cdots j_k)} = q(2k+n) J^{(i_1}{}_r Q_q^{i_2 \cdots i_{k+n} j_1 \cdots j_k) r} \;, \tag{4.11}$$

with the same charge q for all the component superfields. This leads to

$$J\, Q_q^{(n)}(z,u) = q\Big(J^{--}D^{++} - J^{++}D^{--} + nJ^{+-}\Big) Q_q^{(n)}(z,u) \;. \tag{4.12}$$

The J-charge turns out to be uniquely fixed, $q = 1$, if $Q_q^{(n)}$ is covariantly analytic, $\mathcal{D}^+_{\hat{\alpha}} Q_q^{(n)} = 0$.

To summarise, given a covariantly analytic superfield $Q^{(n)}(z,u)$,

$$\mathcal{D}^+_{\hat{\alpha}} Q^{(n)} = 0 \;, \tag{4.13}$$

the infinitesimal isometry transformation acts on it as follows:

$$\begin{aligned} \delta_\xi Q^{(n)} &= -\Big(\xi + \mathrm{i}\,\rho J\Big)\, Q^{(n)} \\ &= -\Big(\xi^{\hat{a}}\mathcal{D}_{\hat{a}} - \xi^{+\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}} + \mathrm{i}\,\rho\big(J^{--}D^{++} - J^{++}D^{--} + nJ^{+-}\big)\Big) Q^{(n)} \;. \end{aligned} \tag{4.14}$$

Given two covariantly analytic superfields $Q^{(n)}$ and $Q^{(m)}$, their product $Q^{(n)}Q^{(m)}$ is covariantly analytic and transforms as $Q^{(n+m)}$. In addition, the superfield $D^{++}Q^{(n)}$ can be seen to be covariantly analytic and transform as $Q^{(n+2)}$.

4.2 Harmonic action principle

After having introduced various analytic multiplets in $\mathrm{AdS}^{5|8} \times S^2$, let us turn to constructing a supersymmetric action. It is worth recalling that in the flat global case ($\omega = 0$), the action principle in 5D harmonic superspace naturally generalizes the original 4D action rule [1, 2] and is given by [10]

$$\int \mathrm{d}^5 x \int \mathrm{d}u \, (\hat{D}^-)^4 \, L^{(+4)} \Big| \,, \qquad (\hat{D}^-)^4 = -\frac{1}{96} \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} D^-_{\hat{\alpha}} D^-_{\hat{\beta}} D^-_{\hat{\gamma}} D^-_{\hat{\delta}} \,, \tag{4.15}$$

where $D^-_{\hat{\alpha}} = D^i_{\hat{\alpha}} u^-_i$, $D^i_{\hat{\alpha}}$ are the flat covariant derivatives, and $L^{(+4)}$ is a real analytic Lagrangian of harmonic charge +4, $D^+_{\hat{\alpha}} L^{(+4)} = 0$.

We would like to generalize the flat action to the case of $\mathrm{AdS}^{5|8}$ using the following ansatz:

$$\begin{aligned} S &= S_0 + a_1 S_1 + a_2 S_2 \\ &= \int \mathrm{d}^5 x \, e \int \mathrm{d}u \, \Big[(\hat{\mathcal{D}}^-)^4 + a_1 \, \omega J^{--} (\hat{\mathcal{D}}^-)^2 + a_2 \, (\omega J^{--})^2 \Big] \mathcal{L}^{(+4)} \Big| \,, \end{aligned} \tag{4.16}$$

where $\mathcal{L}^{(+4)}$ is now covariantly analytic, $\mathcal{D}^+_{\hat{\alpha}} \mathcal{L}^{(+4)} = 0$,

$$(\hat{\mathcal{D}}^-)^2 = \mathcal{D}^{-\hat{\alpha}} \mathcal{D}^-_{\hat{\alpha}} \,, \qquad (\hat{\mathcal{D}}^-)^4 = -\frac{1}{96} \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \mathcal{D}^-_{\hat{\alpha}} \mathcal{D}^-_{\hat{\beta}} \mathcal{D}^-_{\hat{\gamma}} \mathcal{D}^-_{\hat{\delta}} \,, \tag{4.17}$$

and a_1, a_2 are two constants to be determined. It is assumed that the above action is evaluated in Wess-Zumino gauge (2.31), using the bar projection (2.32), and as usual e stands for the determinant of the vielbein, $e = \det(e_{\hat{m}}{}^{\hat{a}})$, with $e_{\hat{m}}{}^{\hat{a}} e_{\hat{a}}{}^{\hat{n}} = \delta_{\hat{m}}{}^{\hat{n}}$.

In accordance with the definition of S, there are several rules for integration by parts which one can use in practice:

$$\int \mathrm{d}^5 x \, e \int \mathrm{d}u \, \mathcal{D}_{\hat{a}} Q^{\hat{a}} | = 0 \,, \tag{4.18}$$

$$\int \mathrm{d}^5 x \, e \int \mathrm{d}u \, D^{++} Q^{--} | = \int \mathrm{d}^5 x \, e \int \mathrm{d}u \, D^{--} Q^{++} | = 0 \,, \tag{4.19}$$

$$\int \mathrm{d}^5 x \, e \int \mathrm{d}u \, J \, Q^{(0)} | = 0 \,. \tag{4.20}$$

Here $Q^{(0)}$ is a covariantly analytic superfield of harmonic charge 0.

Our aim is to find the constants a_1, a_2 for which S is invariant under the isometry transformations of $\mathrm{AdS}^{5|8}$. Let us first compute the variation of S_0 under infinitesimal isometry transformations. Due to (3.1), we have

$$\begin{aligned} \delta\Big((\hat{\mathcal{D}}^-)^4 \mathcal{L}^{(+4)} \Big) &= (\hat{\mathcal{D}}^-)^4 \delta \mathcal{L}^{(+4)} = -(\hat{\mathcal{D}}^-)^4 \Big(\xi + \mathrm{i} \, \rho J \Big) \mathcal{L}^{(+4)} \\ &= -\Big(\xi + \mathrm{i} \, \rho J + \Lambda^{\hat{\alpha}\hat{\beta}} M_{\hat{\alpha}\hat{\beta}} \Big) (\hat{\mathcal{D}}^-)^4 \mathcal{L}^{(+4)} = -\Big(\xi + \mathrm{i} \, \rho J \Big) (\hat{\mathcal{D}}^-)^4 \mathcal{L}^{(+4)} \,. \end{aligned} \tag{4.21}$$

Since $\mathcal{L}^{(+4)}$ is covariantly analytic, we obtain

$$\begin{aligned}\delta_\xi S_0 &= -\int \mathrm{d}^5x\, e \int \mathrm{d}u\, \Big(\xi + \mathrm{i}\,\rho J\Big)(\hat{\mathcal{D}}^-)^4 \mathcal{L}^{(+4)}\Big| \\ &= \int \mathrm{d}^5x\, e \int \mathrm{d}u\, \Big(\xi^{+\hat\alpha}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^4 - \xi^{-\hat\alpha}[\mathcal{D}^+_{\hat\alpha}, (\hat{\mathcal{D}}^-)^4]\Big)\mathcal{L}^{(+4)}\Big|\ . \end{aligned} \tag{4.22}$$

Here we have also used eqs. (4.18) and (4.20).

To compute $\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^4$ in (4.22), we observe that $\mathcal{D}^-_{[\hat\alpha}\mathcal{D}^-_{\hat\beta}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}\mathcal{D}^-_{\hat\rho]} = 0$, and then

$$\begin{aligned} 0 &= 5\varepsilon^{\hat\beta\hat\gamma\hat\delta\hat\rho}\mathcal{D}^-_{[\hat\alpha}\mathcal{D}^-_{\hat\beta}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}\mathcal{D}^-_{\hat\rho]} \\ &= \varepsilon^{\hat\beta\hat\gamma\hat\delta\hat\rho}\Big[\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}\mathcal{D}^-_{\hat\rho} + \mathcal{D}^-_{\hat\rho}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta} + \mathcal{D}^-_{\hat\delta}\mathcal{D}^-_{\hat\rho}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}\mathcal{D}^-_{\hat\gamma} \\ &\qquad + \mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}\mathcal{D}^-_{\hat\rho}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} + \mathcal{D}^-_{\hat\beta}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}\mathcal{D}^-_{\hat\rho}\mathcal{D}^-_{\hat\alpha}\Big]\ . \end{aligned} \tag{4.23}$$

Moving $\mathcal{D}^-_{\hat\alpha}$ in each term to the left gives

$$\begin{aligned}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^4 = \frac{\omega J^{--}}{120}\,\varepsilon^{\hat\beta\hat\gamma\hat\delta\hat\rho}\Big[&4\,M_{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}\mathcal{D}^-_{\hat\rho} + 3\,\mathcal{D}^-_{\hat\gamma}M_{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\delta}\mathcal{D}^-_{\hat\rho} \\ &+ 2\,\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}M_{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\rho} + \mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}\mathcal{D}^-_{\hat\rho}M_{\hat\alpha\hat\beta}\Big]\ . \end{aligned} \tag{4.24}$$

This can be further transformed by moving all the Lorentz generators to the right and factors of $\mathcal{D}^-_{\hat\alpha}$ to the left using iteratively the algebra of covariant derivatives. We end up with

$$\begin{aligned}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^4 =\ & \mathcal{D}^-_{\hat\alpha}\Big\{-\frac{5}{12}\,\omega J^{--}(\hat{\mathcal{D}}^-)^2 + 3\,(\omega J^{--})^2\Big\} \\ & - \Big(\frac{1}{8}\,\omega J^{--}\mathcal{D}^{-\hat\beta}(\hat{\mathcal{D}}^-)^2 - \frac{8}{3}\,(\omega J^{--})^2\mathcal{D}^{-\hat\beta} + \frac{1}{2}\,(\omega J^{--})^2\mathcal{D}^-_{\hat\gamma}M^{\hat\gamma\hat\beta}\Big)M_{\hat\beta\hat\alpha}\ . \end{aligned} \tag{4.25}$$

The expression in the second line does not contribute when acting on a Lorentz scalar such as $\mathcal{L}^{(+4)}$.

To compute $[\mathcal{D}^+_{\hat\alpha}, (\hat{\mathcal{D}}^-)^4]$ in (4.22), we should iteratively use the algebra of covariant derivatives. This is an obvious but tedious procedure. The result is:

$$\begin{aligned}[]
[\mathcal{D}^+_{\hat\alpha}, (\hat{\mathcal{D}}^-)^4] =\ & \frac{1}{4}\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\mathcal{D}^{-\hat\beta}(\hat{\mathcal{D}}^-)^2 + \frac{3}{8}\omega J\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 - \frac{1}{3}\omega J^{--}\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\mathcal{D}^{-\hat\beta} \\
& - \frac{13}{2}\omega^2 J^{--}J\mathcal{D}^-_{\hat\alpha} + \frac{1}{4}\omega J^{+-}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 - 5\omega^2 J^{--}J^{+-}\mathcal{D}^-_{\hat\alpha} \\
& - \frac{1}{8}\omega J^{--}(\hat{\mathcal{D}}^-)^2\mathcal{D}^+_{\hat\alpha} + \frac{3}{8}\omega J^{--}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}\mathcal{D}^{+\hat\beta} - \frac{3}{8}\omega J^{--}\mathcal{D}^-_{\hat\beta}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^{+\hat\beta} \\
& + \frac{55}{12}(\omega J^{--})^2\mathcal{D}^+_{\hat\alpha} + \omega J^{--}\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\gamma}M^{\hat\beta\hat\gamma} - \frac{3}{2}\omega^2 J^{--}J\mathcal{D}^{-\hat\beta}M_{\hat\beta\hat\alpha} \\
& + 2\omega^2 J^{--}J^{+-}\mathcal{D}^{-\hat\beta}M_{\hat\beta\hat\alpha} - \frac{1}{4}\omega J^{+-}\mathcal{D}^{-\hat\beta}(\hat{\mathcal{D}}^-)^2 M_{\hat\beta\hat\alpha} \\
& - \omega^2 J^{--}J^{+-}\mathcal{D}^-_{\hat\gamma}M^{\hat\beta\hat\gamma}M_{\hat\beta\hat\alpha}\ .
\end{aligned} \tag{4.26}$$

Using the relations (4.25) and (4.26), and also the integration by parts identities (4.18) and (4.20), variation (4.22) turns into

$$
\begin{aligned}
\delta_\xi S_0 = &\int \mathrm{d}^5x\, e \int \mathrm{d}u\, \xi^{+\hat\alpha}\Big[-\frac{5}{12}\,\omega J^{--}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 + 3\,(\omega J^{--})^2\mathcal{D}^-_{\hat\alpha}\Big]\mathcal{L}^{(+4)}\Big| \\
&+ \int \mathrm{d}^5x\, e \int \mathrm{d}u\, \Big[-\frac{1}{4}(\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\xi^{-\hat\beta})\mathcal{D}^{-\hat\alpha}(\hat{\mathcal{D}}^-)^2 + \frac{3}{8}\omega(J\xi^{-\hat\alpha})\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 \\
&\qquad + \frac{1}{3}\omega J^{--}(\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\xi^{-\hat\beta})\mathcal{D}^{-\hat\alpha} - \frac{13}{2}\omega^2 J^{--}(J\xi^{-\hat\alpha})\mathcal{D}^-_{\hat\alpha} \\
&\qquad - \frac{1}{4}\omega J^{+-}\xi^{-\hat\alpha}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 + 5\omega^2 J^{--}J^{+-}\xi^{-\hat\alpha}\mathcal{D}^-_{\hat\alpha}\Big]\mathcal{L}^{(+4)}\Big| \;. \qquad (4.27)
\end{aligned}
$$

Finally, it remains to note $J\xi^-_{\hat\alpha} = J^{--}\xi^+_{\hat\alpha} - J^{+-}\xi^-_{\hat\alpha}$, and also make use of eq. (3.14) projected to the minus-harmonics

$$
\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\,\xi^{-\hat\beta} = -\frac{5}{2}\,\omega(J^{--}\xi^+_{\hat\alpha} - J^{+-}\xi^-_{\hat\alpha}) \;. \qquad (4.28)
$$

As a result, the variation of S_0 under the isometry transformations takes the final form:

$$
\begin{aligned}
\delta_\xi S_0 = \int \mathrm{d}^5x\, e \int \mathrm{d}u\, \Big[&-\frac{2}{3}\,\omega J^{--}\xi^{+\hat\alpha}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 - \frac{8}{3}\,(\omega J^{--})^2\xi^{+\hat\alpha}\mathcal{D}^-_{\hat\alpha} \\
&+ \frac{32}{3}\,\omega^2 J^{--}J^{+-}\xi^{-\hat\alpha}\mathcal{D}^-_{\hat\alpha}\Big]\mathcal{L}^{(+4)}\Big| \;. \qquad (4.29)
\end{aligned}
$$

The next step is to compute the variation of the functional S_1 appearing in our action (4.16). Here the procedure is the same as for S_0. Varying

$$
\delta\Big((\hat{\mathcal{D}}^-)^2\mathcal{L}^{(+4)}\Big) = -(\hat{\mathcal{D}}^-)^2\Big(\xi + \mathrm{i}\,\rho J\Big)\mathcal{L}^{(+4)} = -\Big(\xi + \mathrm{i}\,\rho J\Big)(\hat{\mathcal{D}}^-)^2\mathcal{L}^{(+4)} \;, \qquad (4.30)
$$

we get

$$
\delta_\xi S_1 = \int \mathrm{d}^5x\, e \int \mathrm{d}u\, \Big(\omega J^{--}\xi^{+\hat\alpha}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 - \omega J^{--}\xi^{+\hat\alpha}[\mathcal{D}^+_{\hat\alpha}, (\hat{\mathcal{D}}^-)^2]\Big)\mathcal{L}^{(+4)}\Big| \;. \qquad (4.31)
$$

Using the algebra of covariant derivatives gives

$$
\begin{aligned}
[\mathcal{D}^+_{\hat\alpha}, (\hat{\mathcal{D}}^-)^2] = &-4\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\mathcal{D}^{-\hat\beta} - 6\omega J\mathcal{D}^-_{\hat\alpha} + 12\omega J^{+-}\mathcal{D}^-_{\hat\alpha} \\
&-2\omega J^{--}\mathcal{D}^+_{\hat\alpha} + 8\omega J^{+-}\mathcal{D}^{-\hat\beta}M_{\hat\beta\hat\alpha} \;. \qquad (4.32)
\end{aligned}
$$

As a result, the variation of S_1 is

$$
\begin{aligned}
\delta_\xi S_1 = \int \mathrm{d}^5x\, e \int \mathrm{d}u\, \Big[&\omega J^{--}\xi^{+\hat\alpha}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 + 4(\omega J^{--})^2\xi^{+\hat\alpha}\mathcal{D}^-_{\hat\alpha} \\
&- 16\omega^2 J^{--}J^{+-}\xi^{-\hat\alpha}\mathcal{D}^-_{\hat\alpha}\Big]\mathcal{L}^{(+4)}\Big| \;. \qquad (4.33)
\end{aligned}
$$

It is seen that (4.33) is proportional to (4.29). Therefore, our ansatz (4.16) leads to the unique supersymmetric action: $a_1 = 2/3$ and $a_2 = 0$.

The supersymmetric action is

$$S = \int \mathrm{d}^5x\, e \int \mathrm{d}u \left\{ (\hat{\mathcal{D}}^-)^4 + \frac{2}{3}\, \omega\, J^{--} (\hat{\mathcal{D}}^-)^2 \right\} \mathcal{L}^{(+4)} \Big| \ , \qquad \mathcal{D}^+_{\hat{\alpha}} \mathcal{L}^{(+4)} = 0 \ . \quad (4.34)$$

This is the main result of this section.

By construction, the Lagrangian in (4.34) is a covariantly analytic superfield of harmonic charge +4. It should be also chosen to be real with respect to analyticity preserving conjugation [1] (see also subsection 5.1), and then action (4.34) can be seen to be real. Otherwise, $\mathcal{L}^{(+4)}$ is completely arbitrary. Therefore, a great many flat superspace actions [2] can be lifted to the AdS superspace. For instance, an off-shell hypermultiplet can be realized in terms of a covariantly analytic superfield $q^+(z, u)$ and its conjugate $\tilde{q}^+(z, u)$, with respect to the anlyticity preserving conjugation. To describe its dynamics, one can choose

$$\mathcal{L}^{(+4)} = -\tilde{q}^+ D^{++} q^+ + \lambda\, (\tilde{q}^+ q^+)^2 \ , \quad (4.35)$$

with λ a coupling constant.

5 Projective superspace approach

In the projective superspace approach to d-dimensional theories with eight supercharges, one deals with superfields that live in $\mathcal{M}^{d|8} \times S^2$, where $\mathcal{M}^{d|8}$ denotes the conventional superspace, $d \leq 6$, and S^2 the two-sphere. Such superfields are required to (i) be Grassmann analytic, i.e. to be annihilated by one half of the supercharges; (ii) be holomorphic on an open domain of S^2. The latter requirement is equivalently achieved by considering superfields $\Psi^{(n)}(z, u^+)$ which are holomorphic functions of a single isotwsitor $u^{+i} \in \mathbb{C}^2 - \{0\}$, and have definite degree of homogeneity with respect to u^+, $\Psi^{(n)}(z, c\, u^+) = c^n\, \Psi^{(n)}(z, u^+)$. The variables u^{+i} can be viewed as homogeneous coordinates for $\mathbb{C}P^1$. A second linearly independent isotwistor, u^{-i}, is only required (as a purely auxiliary means, without any intrinsic significance) for constructing a supersymmetric action which was proposed originally in four dimensions in [3] and then reformulated in [4] in terms of the projective isotwistor u^{+i}. The terminology "isotwistor" is due to [35, 36].

In the flat global case, the 5D $\mathcal{N} = 1$ extension of the 4D $\mathcal{N} = 2$ supersymmetric

action [4] is as follows[5] [25]:

$$-\frac{1}{2\pi}\oint\frac{u_i^+\mathrm{d}u^{+i}}{(u^+u^-)^4}\int\mathrm{d}^5x\,(\hat{D}^-)^4L^{++}(z,u^+)\Big|\ , \tag{5.1}$$

where

$$D^+_{\hat\alpha}L^{++}(z,u^+)\ =\ 0\ ,\qquad L^{++}(z,c\,u^+)\ =\ c^2\,L^{++}(z,u^+)\ ,\qquad c\in\mathbb{C}^*\ . \tag{5.2}$$

The action is invariant under arbitrary projective transformations of the form

$$({u_i}^-\,,\,{u_i}^+)\ \to\ ({u_i}^-\,,\,{u_i}^+)\,R\ ,\qquad R\ =\ \begin{pmatrix} a & 0\\ b & c\end{pmatrix}\ \in\ \mathrm{GL}(2,\mathbb{C})\ . \tag{5.3}$$

This gauge-like symmetry implies that the action is actually independent of u_i^-. It can be fixed by imposing, for instance, the gauge

$$\begin{aligned} u^{+i}\ \sim\ (1,\zeta)\ =\ \zeta^i\quad &\longrightarrow\quad u_i^+\ \sim\ (-\zeta,1)\ =\ \zeta_i\ ,\\ u^{-i}\ \sim\ (0,-1)\quad &\longrightarrow\quad u_i^-\ \sim\ (1,0)\ , \end{aligned} \tag{5.4}$$

in which the action (5.1) reduces to the standard 5D $\mathcal{N}=1$ projective superspace action [10, 25].

5.1 Projective multiplets

Here we introduce several off-shell projective multiplets that are most interesting from the point of view of model building. By definition, a projective superfield $Q^{(n)}(z,u^+)$ lives on the anti-de Sitter superspace and depends parametrically on a non-vanishing isotwistor $u^{+i}\neq 0$. It is defined to be analytic,

$$\mathcal{D}^+_{\hat\alpha}Q^{(n)}=0\ , \tag{5.5}$$

and transform by the rule

$$\delta Q^{(n)}=-\big(\xi+\mathrm{i}\,\rho J\big)Q^{(n)} \tag{5.6}$$

under the isometry group. We specify J to act on $Q^{(n)}$ as follows

$$J\,Q^{(n)}=-\frac{J^{++}D^{--}Q^{(n)}-nJ^{+-}Q^{(n)}}{(u^+u^-)}\ . \tag{5.7}$$

[5] Note that the action given in eq. (B.1) of [25] contains a wrong overal factor of $\sqrt{-1}$.

This definition involves an external isotwistor u_i^- subject to the only requirement

$$(u^+u^-) \neq 0 \ . \tag{5.8}$$

Since $Q^{(n)}$ is independent of u^-, it is natural to require $J\,Q^{(n)}$ to be independent of u^- as well, that is

$$J^{++}\frac{\partial}{\partial u^{+i}}Q^{(n)} - nJ_{ij}u^{+j}Q^{(n)} = u_i^+ J\,Q^{(n)} \ . \tag{5.9}$$

Contracting this with u^{+i} gives

$$J^{++}u^{+i}\frac{\partial}{\partial u^{+i}}Q^{(n)} = nJ^{++}Q^{(n)} \ . \tag{5.10}$$

Therefore, $Q^{(n)}$ is a homogeneous function of u^+ of degree n,

$$Q^{(n)}(z, c\,u^+) \ = \ c^n\, Q^{(n)}(z, u^+) \ , \qquad c \in \mathbb{C}^* \ . \tag{5.11}$$

The $Q^{(n)}$ will be called a projective superfield of weight n.

As is obvious, the complex conjugate of an analytic superfield is not analytic. However, one can introduce a generalized, analyticity-preserving conjugation [7, 1, 3], $u^{+i} \to \widetilde{u^{+i}}$, which is obtained by composing the complex conjugation, $u^{+i} \to \overline{u^{+i}}$, with the antipodal map $\overline{u^{+i}} \to -u_i^+$. In what follows, it is called "smile-conjugation." It is thus defined to act on the isotwistor $u^+ = (u^{+i})$ by the rule[6]

$$u^+ \ \to \ \tilde{u}^+ = \mathrm{i}\,\sigma_2\, u^+ \ , \qquad \widetilde{\widetilde{(u^{+i})}} = -u^{+i} \ , \tag{5.12}$$

with σ_2 the second Pauli matrix. Its action on the projective superfields is defined to be

$$Q^{(n)}(u^+) \to \widetilde{Q}^{(n)}(u^+) \equiv \bar{Q}^{(n)}\big(\tilde{u}^+\big) \ , \qquad \widetilde{\widetilde{Q}}^{(n)} \ = \ (-1)^n Q^{(n)} \ . \tag{5.13}$$

It is clear that $\widetilde{Q}^{(n)}(u^+)$ is a homogeneous function of u^+ of degree n, that is $\widetilde{Q}^{(n)}(c\,u^+) = c^n\,\widetilde{Q}^{(n)}(u^+)$, with $c \in \mathbb{C}^*$. Due to the identity

$$\widetilde{\mathcal{D}^+_{\hat{\alpha}} Q^{(n)}} \ = \ (-1)^{\epsilon(Q^{(n)})}\, \mathcal{D}^{+\hat{\alpha}} \widetilde{Q}^{(n)} \ , \tag{5.14}$$

the smile-conjugation indeed preserves analyticity.

It is important to note that, in accordance with (5.13), for an even integer weight, $n = 2p$, one can consistently define real projective superfields $R^{(2p)}$ with respect to the smile-conjugation: $\widetilde{R}^{(2p)} = R^{(2p)}$.

[6]Due to projective invariance, $u^{+i} \sim c\,u^{+i}$, the smile-conjugation could be also defined as $u^+ \ \to \ \tilde{u}^+ = -\mathrm{i}\,\sigma_2\, u^+$, instead of (5.12).

Now, let us show that the smile-conjugation is compatible with the superfield transformation law (5.6). To evaluate the smile-conjugate of $J\,Q^{(n)}$, eq. (5.7), we conventionally define the operation of smile-conjugate for $u^- = (u^{-i})$ to be identical to that we have already chosen for the isotwistor u^+, that is

$$u^- \;\to\; \tilde{u}^- = \mathrm{i}\,\sigma_2\,u^- \;, \qquad \widetilde{\widetilde{(u^{-i})}} = -u^{-i} \;, \tag{5.15}$$

We should emphasize that such a definition is completely conventional in the sense that the projective superfields are independent of the isotwistor u^-. Then it holds

$$\widetilde{D^{--}} = D^{--} \;, \quad \widetilde{J^{\pm\pm}} \;=\; -J^{\pm\pm} \;, \quad \widetilde{J^{+-}} \;=\; -J^{+-} \;, \quad \widetilde{(u^+u^-)} \;=\; (u^+u^-) \;. \tag{5.16}$$

This implies

$$\widetilde{J\,Q^{(n)}} \;=\; -\,J\,\widetilde{Q}^{(n)} \;, \tag{5.17}$$

and the smile-conjugate of the transformation law (5.6) is

$$\widetilde{\delta Q^{(n)}} \;=\; -(\xi + \mathrm{i}\,\rho J)\,\widetilde{Q}^{(n)} = \delta\,\widetilde{Q}^{(n)} \;. \tag{5.18}$$

Therefore, the smile-conjugation preserves the superfield transformation laws under the isometry group.

As is known, the space $\mathbb{C}P^1$ can be covered by two charts that are defined in terms of $u^+ = (u^{+\underline{1}}, u^{+\underline{2}})$ as follows: (i) the north chart on which $u^{+\underline{1}} \neq 0$; (ii) the south chart on which $u^{+\underline{2}} \neq 0$. As will be described below, the projective action involves the line integral over a closed contour in $\mathbb{C}P^1$, and this contour can be chosen to lie inside one of the coordinate charts. The latter can be chosen to be the north chart, and that is why our local considerations will be mainly restricted to that chart. In the north chart, we can introduce a projective invariant complex coordinate ζ defined as $u^{+i} = u^{+\underline{1}}\,(1,\zeta)$, with $\zeta = u^{+\underline{2}}/u^{+\underline{1}}$. Since $\tilde{u}^{+i} = (u^{+\underline{2}}, -u^{+\underline{1}})$, the smile-conjugation acts as follows:

$$\zeta \;\to\; -\frac{1}{\zeta}\;. \tag{5.19}$$

The simplest solution to eq. (5.11) is the $O(n)$ multiplet defined by eqs. (4.2) and (4.3). This multiplet is globally defined on $\mathbb{C}P^1$. Allowing for singularities at some points in $\mathbb{C}P^1$ offers the possibility to generate many more interesting supermultiplets. For example, a charged hypermultiplet is described by a weight-one projective superfield $\Upsilon^+(u^+)$ being

holomorphic on $\mathbb{C}P^1 - \{N\}$, where the North pole is identified with $u^{+i} \sim (0,1)$. We can represent $\Upsilon^+(u^+)$ as

$$\Upsilon^+(u^+) = u^{+\underline{1}}\,\Upsilon^+(u^{+i}/u^{+\underline{1}}) \equiv u^{+\underline{1}}\,\Upsilon(\zeta)\ , \qquad \Upsilon(z,\zeta) = \sum_{k=0}^{+\infty} \Upsilon_k(z)\zeta^k\ . \tag{5.20}$$

Its smile-conjugate $\widetilde{\Upsilon}^+(u^+)$ is holomorphic on $\mathbb{C}P^1 - \{S\}$, where the South pole is identified with $u^{+i} \sim (1,0)$. We can represent $\widetilde{\Upsilon}^+(u^+)$ as

$$\widetilde{\Upsilon}^+(u^+) = u^{+\underline{2}}\,\widetilde{\Upsilon}^+(u^{+i}/u^{+\underline{2}}) \equiv u^{+\underline{2}}\,\widetilde{\Upsilon}(\zeta)\ , \qquad \widetilde{\Upsilon}(z,\zeta) = \sum_{k=0}^{+\infty} \bar{\Upsilon}_k(z)\,\frac{(-1)^k}{\zeta^k}\ , \tag{5.21}$$

with $\bar{\Upsilon}_k(z)$ the complex conjugate of $\Upsilon_k(z)$. To describe an off-shell vector multiplet, one should use a real weight-zero projective superfield $V(u^+)$ being holomorphic on $\mathbb{C}P^1 - \{N \cup S\}$. It can be represented as

$$V(z,\zeta) = \sum_{k=-\infty}^{+\infty} V_k(z)\zeta^k\ , \qquad \bar{V}_k = (-1)^k V_{-k}\ . \tag{5.22}$$

5.2 Projective action principle

Our aim here is to find a generalization of the flat superspace action (5.1) to the case of $\mathrm{AdS}^{5|8}$ superspace. We start with the following ansatz[7]

$$\begin{aligned} S &= S_0 + \beta_1\, S_1 + \beta_2\, S_2 \\ &= -\frac{1}{2\pi}\oint \frac{u_i^+ \mathrm{d}u^{+i}}{(u^+u^-)^4}\int \mathrm{d}^5x\, e\Big[(\hat{\mathcal{D}}^-)^4 + \beta_1\omega J^{--}(\hat{\mathcal{D}}^-)^2 + \beta_2(\omega J^{--})^2\Big]\mathcal{L}^{++}(z,u^+)\Big|\ . \end{aligned} \tag{5.23}$$

Here $\mathcal{L}^{++}(z,u^+)$ is a covariantly analytic superfield, $\mathcal{D}^+_{\hat{\alpha}}\mathcal{L}^{++} = 0$, which is homogeneous in u_i^+ of degree $+2$. The line integral in (5.23) is carried out over a closed contour, $\gamma = \{u_i^+(t)\}$, in the space of u^+ variables. The integrand in (5.23) involves a constant (i.e. time-independent) isotwistor u_i^- subject to the only condition that $u^+(t)$ and u^- form a linearly independent basis at each point of the contour γ, that is $(u^+u^-) \neq 0$.

Our first requirement is that the action (5.23) be invariant under the projective gauge transformations (5.3). First of all, it is obvious that (5.23) is invariant under arbitrary

[7] An alternative approach to introduce the projective action consists in using a proper generalization of the procedure given in [37]. The latter allows one to derive the projective action as a singular limit of the harmonic action.

scale transformations $u_i^+(t) \rightarrow c(t)\, u_i^+(t)$, with $c(t) \neq 0$. It is thus only necessary to analyse projective transformations of u^- of the form

$$u_i^- \;\rightarrow\; \tilde{u}_i^- \;=\; a(t)\, u_i^- + b(t)\, u_i^+(t) \;, \qquad a(t) \neq 0 \;. \tag{5.24}$$

Since both u^- and $\tilde{u}^-$ should be time independent, the coefficients should obey the equations (using the notation $\overset{\bullet}{f} \equiv \mathrm{d}f(t)/\mathrm{d}t$, for a function $f(t)$):

$$\overset{\bullet}{a} = b\, \frac{(\overset{\bullet}{u}{}^+ u^+)}{(u^+u^-)} \;, \qquad \overset{\bullet}{b} = -b\, \frac{(\overset{\bullet}{u}{}^+ u^-)}{(u^+u^-)} \;. \tag{5.25}$$

As is obvious, the action (5.23) is invariant under arbitrary scale transformations $u_i^- \rightarrow a(t)\, u_i^-$, with $a \neq 0$. Therefore, it only remains to analyse infinitesimal transformations of the form $\delta u_i^- = b(t) u_i^+$, with $b(t)$ obeying the differential equation (5.25). This transformation induces the following variations:

$$\delta \mathcal{D}^-_{\hat\alpha} \;=\; b\, \mathcal{D}^+_{\hat\alpha} \;, \qquad \delta J^{--} \;=\; 2b\, J^{+-} \;. \tag{5.26}$$

Let us start by evaluating the variation of S_0. Using the condensed notation

$$\mathrm{d}\mu^{++} \equiv -\frac{1}{2\pi} \frac{u_i^+ \mathrm{d}u^{+i}}{(u^+u^-)^4} = -\frac{1}{2\pi} \frac{(\overset{\bullet}{u}{}^+ u^+)}{(u^+u^-)^4}\, \mathrm{d}t \;, \tag{5.27}$$

we obtain

$$\begin{aligned} \delta S_0 \;=&\; \oint \mathrm{d}\mu^{++} \int \mathrm{d}^5x\, e\, \Big[\delta(\hat{\mathcal{D}}^-)^4\Big] \mathcal{L}^{++}\Big| \\ =&\; -\frac{\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}}{96} \oint \mathrm{d}\mu^{++}\, b \int \mathrm{d}^5x\, e\, \Big[\, 3\, \mathcal{D}^-_{\hat\alpha} \mathcal{D}^-_{\hat\beta} \{\mathcal{D}^+_{\hat\gamma}, \mathcal{D}^-_{\hat\delta}\} + 2\, \mathcal{D}^-_{\hat\alpha} \{\mathcal{D}^+_{\hat\beta}, \mathcal{D}^-_{\hat\gamma}\} \mathcal{D}^-_{\hat\delta} \\ &\qquad\qquad + \{\mathcal{D}^+_{\hat\alpha}, \mathcal{D}^-_{\hat\beta}\} \mathcal{D}^-_{\hat\gamma} \mathcal{D}^-_{\hat\delta}\, \Big] \mathcal{L}^{++}\Big| \;. \end{aligned} \tag{5.28}$$

Now, making use of the covariant derivatives algebra (2.28a–2.29b) and the identities

$$[J, \mathcal{D}^+_{\hat\alpha}] = \frac{1}{(u^+u^-)} \left(J^{+-} \mathcal{D}^+_{\hat\alpha} - J^{++} \mathcal{D}^-_{\hat\alpha} \right) \;, \tag{5.29a}$$

$$[J, \mathcal{D}^-_{\hat\alpha}] = \frac{1}{(u^+u^-)} \left(J^{--} \mathcal{D}^+_{\hat\alpha} - J^{+-} \mathcal{D}^-_{\hat\alpha} \right) \;, \tag{5.29b}$$

we can systematically move in (5.28) all space-time derivatives to the left (neglecting total space-time derivatives) and the J operator to the right. This gives

$$\begin{aligned} \delta S_0 \;=&\; \oint \mathrm{d}\mu^{++}\, b \int \mathrm{d}^5x\, e\, \Big[-\frac{7}{12} \omega J^{+-} (\hat{\mathcal{D}}^-)^2 - \frac{3}{8} \omega (u^+u^-) (\hat{\mathcal{D}}^-)^2 J \\ &\qquad + \frac{11}{2} (u^+u^-) \omega^2 J^{--} J \Big] \mathcal{L}^{++}\Big| \;. \end{aligned} \tag{5.30}$$

To transform the second and third terms in the square brackets, we should first recall how J acts on the Lagrangian,

$$J\,\mathcal{L}^{++} = \frac{1}{(u^+u^-)}\left(-J^{++}D^{--}\mathcal{L}^{++} + 2J^{+-}\mathcal{L}^{++}\right)\ . \tag{5.31}$$

Since $\mathcal{L}^{++}$ is a homogeneous function of degree two, we also have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^{++} &= \frac{(\dot{u}^+u^-)}{(u^+u^-)}u^{+i}\frac{\partial}{\partial u^{+i}}\mathcal{L}^{++} - \frac{(\dot{u}^+u^+)}{(u^+u^-)}D^{--}\mathcal{L}^{++}\\ &= 2\frac{(\dot{u}^+u^-)}{(u^+u^-)}\mathcal{L}^{++} - \frac{(\dot{u}^+u^+)}{(u^+u^-)}D^{--}\mathcal{L}^{++}\ .\end{aligned} \tag{5.32}$$

The latter results leads to

$$(\dot{u}^+u^+)\,J\,\mathcal{L}^{++} = J^{++}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^{++} - 2\frac{(\dot{u}^+u^-)}{(u^+u^-)}J^{++}\mathcal{L}^{++} + 2\frac{(\dot{u}^+u^+)}{(u^+u^-)}J^{+-}\mathcal{L}^{++}\ . \tag{5.33}$$

One more technical observation,

$$\frac{\mathrm{d}}{\mathrm{d}t}J^{++} = 2\frac{(\dot{u}^+u^-)}{(u^+u^-)}J^{++} - 2\frac{(\dot{u}^+u^+)}{(u^+u^-)}J^{+-}\ , \tag{5.34}$$

allows us to obtain the following identity:

$$b\,\frac{(\dot{u}^+u^+)}{(u^+u^-)^3}\,J\,\mathcal{L}^{++} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{b\,J^{++}}{(u^+u^-)^3}\,\mathcal{L}^{++}\right] + 4b\,\frac{(\dot{u}^+u^+)}{(u^+u^-)^4}J^{+-}\mathcal{L}^{++}\ . \tag{5.35}$$

Then (5.30) becomes

$$\delta S_0 = \oint \mathrm{d}\mu^{++}\,b\int \mathrm{d}^5x\,e\Big[-\frac{25}{12}\,\omega J^{+-}(\hat{\mathcal{D}}^-)^2 + 22\,\omega^2 J^{--}J^{+-}\Big]\mathcal{L}^{++}\Big|\ . \tag{5.36}$$

Using the same procedure, for δS_1 and δS_2 we find

$$\delta S_1 = \oint \mathrm{d}\mu^{++}\,b\int \mathrm{d}^5x\,e\Big[2\,\omega J^{+-}(\hat{\mathcal{D}}^-)^2 + 48\,\omega^2 J^{--}J^{+-}\Big]\mathcal{L}^{++}\Big|\ , \tag{5.37a}$$

$$\delta S_2 = \oint \mathrm{d}\mu^{++}\,b\int \mathrm{d}^5x\,e\,4\,\omega^2 J^{--}J^{+-}\mathcal{L}^{++}\Big|\ . \tag{5.37b}$$

The relations obtained show that the requirement of projective invariance, $\delta S = \delta S_0 + \beta_1\,\delta S_1 + \beta_2\,\delta S_2 = 0$, uniquely fixes the coefficients in in (5.23) as follows: $\beta_1 = 25/24$ and $\beta_2 = -18$. We end up with the projective-invariant action

$$S = -\frac{1}{2\pi}\oint \frac{u_i^+\mathrm{d}u^{+i}}{(u^+u^-)^4}\int \mathrm{d}^5x\,e\,\Big[(\hat{\mathcal{D}}^-)^4 + \frac{25}{24}\,\omega J^{--}(\hat{\mathcal{D}}^-)^2 - 18\,(\omega J^{--})^2\Big]\mathcal{L}^{++}\Big|\ . \tag{5.38}$$

Now, we are going to demonstrate that (5.38) is supersymmetric, that is this action is invariant under the isometry group of $\mathrm{AdS}^{5|8}$. This requires us to carry out calculations that are very similar to those presented in section 4 for the harmonic case. But there are two technical features being specific for the projective case: (i) unlike the harmonic case, we have $(u^+u^-) \neq 1$ in general, and therefore it is necessary to keep track of the factors of (u^+u^-); (ii) unlike the harmonic superspace identity (4.20), in general we have $\oint \mathrm{d}\mu^{++} J\, Q^{--} \neq 0$. In all variations involving the U(1) generator J, we will systematically move J's to the right to hit the Lagrangian $\mathcal{L}^{++}$, so that eqs. (5.31) and (5.33) can be applied.

We start by computing the variation of S_0 under the infinitesimal isometry transformation. Making use of

$$\delta\Big((\hat{\mathcal{D}}^-)^4\mathcal{L}^{++}\Big) = -(\hat{\mathcal{D}}^-)^4\Big(\xi + \mathrm{i}\,\rho J\Big)\mathcal{L}^{++} = -\Big(\xi + \mathrm{i}\,\rho J\Big)(\hat{\mathcal{D}}^-)^4\mathcal{L}^{++} \tag{5.39}$$

gives

$$\begin{aligned}\delta_\xi S_0 = \oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\Big[\frac{1}{(u^+u^-)}\Big(\xi^{+\hat\alpha}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^4 - \xi^{-\hat\alpha}[\mathcal{D}^+_{\hat\alpha},(\hat{\mathcal{D}}^-)^4]\Big)\\ - \mathrm{i}\,\rho\Big([J,(\hat{\mathcal{D}}^-)^4] + (\hat{\mathcal{D}}^-)^4 J\Big)\Big]\mathcal{L}^{++}\Big| \;.\end{aligned} \tag{5.40}$$

To evaluate $\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^4\mathcal{L}^{++}$, we note that eq. (4.25) holds even if $(u^+u^-) \neq 1$, since in the derivation of (4.25) we only used eq. (2.28c) and the commutation relations of the Lorentz generator $M_{\hat\alpha\hat\beta}$ with the covariant derivatives, and both results are clearly not affected by the normalization of (u^+u^-). Therefore, for the first term on the right of (5.40) we have

$$\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^4\mathcal{L}^{++} = \Big[-\frac{5}{12}\,\omega J^{--}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 + 3\,(\omega J^{--})^2\mathcal{D}^-_{\hat\alpha}\Big]\mathcal{L}^{++} \;. \tag{5.41}$$

For the operator $[\mathcal{D}^+_{\hat\alpha},(\hat{\mathcal{D}}^-)^4]$, which appears in (5.40), we have derived eq. (4.26) in the harmonic case. Now, in evaluating the second term on the right of (5.40), we should take care of the factors of (u^+u^-), as well as to move the U(1) generator J to the right. This gives

$$\begin{aligned}[\mathcal{D}^+_{\hat\alpha},(\hat{\mathcal{D}}^-)^4] = \frac{1}{4}(u^+u^-)\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\mathcal{D}^{-\hat\beta}(\hat{\mathcal{D}}^-)^2 + \frac{3}{8}(u^+u^-)\omega\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2 J - \frac{7}{8}\omega J^{+-}\mathcal{D}^-_{\hat\alpha}(\hat{\mathcal{D}}^-)^2\\ - \frac{11}{6}(u^+u^-)\omega J^{--}\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\mathcal{D}^{-\hat\beta} - \frac{17}{4}(u^+u^-)\omega^2 J^{--}\mathcal{D}^-_{\hat\alpha}J + \frac{33}{4}\omega^2 J^{--}J^{+-}\mathcal{D}^-_{\hat\alpha} + \cdots \;,\end{aligned} \tag{5.42}$$

where the dots denote those terms which do not contribute when acting on Lorentz scalar and analytic superfields such as the Lagrangian $\mathcal{L}^{++}$. Inserting (5.42) into δS_0, one can get

read of the terms with vector covariant derivatives by taking into account the integration by parts rule (4.18) and

$$\mathrm{i}\mathcal{D}_{\hat{\alpha}\hat{\beta}}\xi^{-\hat{\beta}} = -\frac{5\omega}{2(u^+u^-)}(J^{--}\xi^+_{\hat{\alpha}} - J^{+-}\xi^-_{\hat{\alpha}}) \ . \tag{5.43}$$

To evaluate the contributions to δS_0 which contain $J\mathcal{L}^{++}$, we note that eqs. (5.33) and (5.34) imply

$$\frac{(\dot{u}^+u^+)}{(u^+u^-)^4}\, J\,\mathcal{L}^{++} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{J^{++}\mathcal{L}^{++}}{(u^+u^-)^4}\Big) + 4\frac{(\dot{u}^+u^+)}{(u^+u^-)^5}\, J^{+-}\mathcal{L}^{++} \ . \tag{5.44}$$

The latter observation tells us

$$\oint \mathrm{d}\mu^{++}\, \mathcal{O}(u^-)\, J\,\mathcal{L}^{++}\Big| = 4\oint \mathrm{d}\mu^{++}\, \frac{J^{+-}}{(u^+u^-)}\, \mathcal{O}(u^-)\,\mathcal{L}^{++}\Big| \ , \tag{5.45}$$

for any operator $\mathcal{O}(u^-)$ independent of u^+. It follows that

$$\begin{aligned} -\oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\,[\mathcal{D}^+_{\hat{\alpha}},(\hat{\mathcal{D}}^-)^4]\mathcal{L}^{++}\Big| = \frac{5}{2}\oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\,\frac{\omega J^{--}}{(u^+u^-)}\Big[-\frac{1}{4}\xi^{+\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}}(\hat{\mathcal{D}}^-)^2 \\ +\frac{11}{6}\omega J^{--}\xi^{+\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}} + \frac{5}{3}\omega J^{+-}\,\xi^{-\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}}\Big]\mathcal{L}^{++}\Big| \ . \end{aligned} \tag{5.46}$$

Completely similar considerations, using also $\mathcal{D}_{\hat{\alpha}\hat{\beta}}\,\rho = 0$ (3.13), give

$$\begin{aligned} -\oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\,\rho[J,(\hat{\mathcal{D}}^-)^4]\mathcal{L}^{++}\Big| = \oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\,\frac{\rho\, J^{+-}}{(u^+u^-)}\Big[\,4(\hat{\mathcal{D}}^-)^4 \\ +\frac{25}{12}\,\omega J^{--}(\hat{\mathcal{D}}^-)^2 - 22\,(\omega J^{--})^2\Big]\mathcal{L}^{++}\Big| \ . \end{aligned} \tag{5.47}$$

As a result, the variation $\delta_\xi S_0$ can be represented in the form

$$\begin{aligned} \delta_\xi S_0 = \oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\Big[-\frac{25}{24}\frac{\omega J^{--}}{(u^+u^-)}\,\xi^{+\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}}(\hat{\mathcal{D}}^-)^2 + \frac{91}{12}\frac{(\omega J^{--})^2}{(u^+u^-)}\,\xi^{+\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}} \\ +\frac{25}{6}\frac{\omega^2 J^{--}J^{+-}}{(u^+u^-)}\,\xi^{-\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}} + \frac{25}{12}\frac{\omega J^{--}J^{+-}}{(u^+u^-)}\,\rho(\hat{\mathcal{D}}^-)^2 - 22\,\frac{(\omega J^{--})^2 J^{+-}}{(u^+u^-)}\,\rho\Big]\mathcal{L}^{++}\Big| \ . \end{aligned} \tag{5.48}$$

The variations $\delta_\xi S_1$ and $\delta_\xi S_2$ can be computed by similar means. The results are:

$$\begin{aligned} \delta_\xi S_1 = \oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\Big[\frac{\omega J^{--}}{(u^+u^-)}\,\xi^{+\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}}(\hat{\mathcal{D}}^-)^2 + 10\,\frac{(\omega J^{--})^2}{(u^+u^-)}\,\xi^{+\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}} \\ -4\,\frac{\omega^2 J^{--}J^{+-}}{(u^+u^-)}\,\xi^{-\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}} - 2\,\frac{\omega J^{--}J^{+-}}{(u^+u^-)}\,\rho(\hat{\mathcal{D}}^-)^2 - 48\,\frac{(\omega J^{--})^2 J^{+-}}{(u^+u^-)}\,\rho\Big]\mathcal{L}^{++}\Big| \ , \end{aligned} \tag{5.49a}$$

$$\delta_\xi S_2 = \oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\Big[\frac{(\omega J^{--})^2}{(u^+u^-)}\,\xi^{+\hat{\alpha}}\mathcal{D}^-_{\hat{\alpha}} - 4\,\frac{(\omega J^{--})^2 J^{+-}}{(u^+u^-)}\,\rho\Big]\mathcal{L}^{++}\Big| \ . \tag{5.49b}$$

Collecting all the results obtained, we conclude

$$\delta_\xi S = 0 \ , \tag{5.50}$$

and therefore the action (5.38) is supersymmetric. Actually, it proves to be the only supersymmetric action in the family (5.23). It is quite remarkable that projective invariance implies supersymmetry and vice versa.

6 Dynamical systems in projective superspace

In this section we study in more detail the projective multiplets and then consider several important supersymmetric theories. To simplify the analysis, it is useful to choose the projective gauge $u^-_{\underline{2}} = 0$. Without loss of generality, one can also work in a representation of the algebra in which $J^{\underline{11}} = J^{\underline{22}} = 0$, and hence $J^{--} = 0$.

6.1 Projective multiplets revisited

In each of the two coordinate charts for $\mathbb{C}P^1$, one can describe the projective multiplets by superfields invariant under the projective transformation (5.11). Let us restrict our consideration to the north chart. Given a complex projective multiplet of weight n, $Q^{(n)}(u^+)$, it can be equivalently described by a holomorphic function $Q^{[n]}(\zeta)$ defined as follows:

$$Q^{(n)}(u^{+i}) = (u^{+\underline{1}})^n Q^{[n]}(\zeta) \ , \qquad Q^{[n]}(\zeta) \equiv Q^{(n)}(1,\zeta) \ . \tag{6.1}$$

Here $Q^{[n]}(\zeta)$ is clearly invariant under (5.11). For the smile-conjugate of $Q^{(n)}(u^+)$, we get

$$\widetilde{Q}^{(n)}(u^{+i}) = (u^{+\underline{2}})^n \widetilde{Q}^{[n]}(\zeta) \ , \qquad \widetilde{Q}^{[n]}(\zeta) = \bar{Q}^{[n]}(-1/\zeta) \ . \tag{6.2}$$

Given a real projective multiplet $R^{(2p)}(u^+)$, with respect to the smile-coinjugation, it can be represented

$$R^{(2p)}(u^+) = ({\rm i}u^{+\underline{1}}u^{+\underline{2}})^p R^{[2p]}(\zeta) \ , \qquad \widetilde{R}^{[2p]}(\zeta) \equiv \widetilde{R}^{[2p]}(-1/\zeta) = R^{[2p]}(\zeta) \ . \tag{6.3}$$

The most general form for $Q^{[n]}(z,\zeta)$ is

$$Q^{[n]}(z,\zeta) = \sum_{k=-\infty}^{+\infty} Q_k^{[n]}(z)\zeta^k \ . \tag{6.4}$$

In the projective gauge chosen ($u_{\underline{2}}^- = 0$, $J^{\underline{11}} = J^{\underline{22}} = 0$), the action of the operator J on our superfield becomes

$$J\, Q^{[n]}(u^+) = -\frac{1}{(u^+u^-)}\left(J^{++}D^{--} - nJ^{+-}\right) Q^{[n]}(u^+) = J^{\underline{12}}\left(n - 2u^{+\underline{2}}\frac{\partial}{\partial u^{+\underline{2}}}\right) Q^{[n]}(u^+) \ .$$

Then, since the isotwistor u^{+i} is neutral under the action of J, it holds

$$\begin{aligned}(u^{+\underline{1}})^n\, J\, Q^{[n]}(\zeta) = J\, Q^{(n)}(u^+) &= J^{\underline{12}}\left(n(u^{+\underline{1}})^n Q^{[n]}(\zeta) - 2u^{+\underline{2}}\frac{\partial}{\partial u^{+\underline{2}}}\Big((u^{+\underline{1}})^n Q^{[n]}(\zeta)\Big)\right) \\ &= (u^{+\underline{1}})^n J^{\underline{12}}\Big(nQ^{[n]}(\zeta) - 2\zeta\frac{\partial}{\partial\zeta}Q^{[n]}(\zeta)\Big) \ ,\end{aligned}$$

and therefore

$$J\, Q^{[n]}(z,\zeta) = J^{\underline{1}}{}_{\underline{1}}\left(2\zeta\frac{\partial}{\partial\zeta} - n\right) Q^{[n]}(z,\zeta) \ , \qquad J\, Q_k^{[n]}(z) = (2k-n)J^{\underline{1}}{}_{\underline{1}}Q_k^{[n]}(z) \ . \quad (6.5)$$

In the case of a real superfield $R^{(2p)}(z,u^+) = (\mathrm{i}u^{+\underline{1}}u^{+\underline{2}})^p R^{[2p]}(z,\zeta)$, we have for $R^{[2p]}$

$$R^{[2p]}(z,\zeta) = \sum_{k=-\infty}^{+\infty} R_k^{[2p]}(z)\zeta^k \ , \qquad \bar{R}_k^{[2p]} = (-1)^k R_{-k}^{[2p]} \ . \quad (6.6)$$

The operator J is represented as follows:

$$J\, R^{[2p]}(z,\zeta) = 2J^{\underline{1}}{}_{\underline{1}}\zeta\frac{\partial}{\partial\zeta}R^{[2p]}(z,\zeta) \ , \qquad J\, R_k^{[2p]}(z) = 2kJ^{\underline{1}}{}_{\underline{1}}R_k^{[2p]}(z) \ . \quad (6.7)$$

Let us analyse the implications of the analyticity condition, $\mathcal{D}^+_{\hat\alpha}Q^{(n)}(u^+) = 0$. It is useful to change the representation for the projective superfields, $Q^{(n)}(u^+) \to Q^{[n]}(\zeta)$. We then have $\mathcal{D}^+_{\hat\alpha}Q^{[n]}(\zeta) = -u^{+\underline{1}}(\zeta\mathcal{D}^{\underline{1}}_{\hat\alpha} - \mathcal{D}^{\underline{2}}_{\hat\alpha})Q^{[n]}(\zeta)$, and therefore the analyticity condition is equivalent to

$$\mathcal{D}^{\underline{2}}_{\hat\alpha}Q^{[n]}(\zeta) = \zeta\mathcal{D}^{\underline{1}}_{\hat\alpha}Q^{[n]}(\zeta) \ . \quad (6.8)$$

For the component superfields $Q_k^{[n]}(z)$, this implies

$$\mathcal{D}^{\underline{2}}_{\hat\alpha}Q_k^{[n]} = \mathcal{D}^{\underline{1}}_{\hat\alpha}Q_{k-1}^{[n]} \ . \quad (6.9)$$

It is natural to think of $\mathcal{D}^{\underline{1}}_{\hat\alpha}$ and $\mathcal{D}^{\underline{2}}_{\hat\alpha}$ as the covariant derivatives associated with two 5D Dirac spinor coordinates, $\theta^{\hat\alpha}_{\underline{1}}$ and their conjugates $\theta^{\hat\alpha}_{\underline{2}}$. It then follows from (6.8) that the dependence of $Q^{[n]}(\zeta)$ on $\theta^{\hat\alpha}_{\underline{2}}$ is completely determined by the dependence of $Q^{[n]}(\zeta)$ on $\theta^{\hat\alpha}_{\underline{1}}$.

Suppose that the expansion of $Q^{[n]}(\zeta)$ in powers of ζ terminates from below

$$Q^{[n]}(z,\zeta) = \sum_{k=L}^{+\infty} Q_k^{[n]}(z)\zeta^k \ . \quad (6.10)$$

Then, eq. (6.9) tells us that the two lowest components of $Q^{[n]}$ are constrained as follows:

$$\begin{aligned} \mathcal{D}^{\underline{2}}_{\hat{\alpha}} Q^{[n]}_L &= 0 \ , \\ (\hat{\mathcal{D}}^{\underline{2}})^2 Q^{[n]}_{L+1} &= 12\omega\, J\, Q^{[n]}_L \ , \end{aligned} \tag{6.11}$$

where

$$(\hat{\mathcal{D}}^i)^2 = \mathcal{D}^{i\hat{\alpha}} \mathcal{D}^i_{\hat{\alpha}} \ , \qquad i = \underline{1}, \underline{2} \ . \tag{6.12}$$

Therefore, $Q^{[n]}_L$ is a five-dimensional *chiral superfield*, while $Q^{[n]}_{L+1}$ a *complex linear superfield*. The union of $Q^{[n]}_L$ and $Q^{[n]}_{L+1}$ forms a 5D analogue of the famous chiral-nonminimal doublet in 4D supersymmetry [38].

Given a real $O(2)$ multiplet $H^{(2)}(z, u^+)$, we can represent $H^{[2]}(z,\zeta)$ in the form

$$H^{[2]}(z,\zeta) = \frac{1}{\zeta}\Phi(z) + G(z) - \zeta\bar{\Phi}(z) \ , \tag{6.13}$$

where Φ is a five-dimensional *chiral superfield*, and G a *real linear superfield*,

$$\begin{aligned} \mathcal{D}^{\underline{2}}_{\hat{\alpha}} \Phi &= 0 \ , \\ (\hat{\mathcal{D}}^{\underline{2}})^2 G &= 0 \ , \qquad \bar{G} = G \ . \end{aligned} \tag{6.14}$$

If the expansion of $Q^{[n]}(\zeta)$ in powers of ζ terminates from above,

$$Q^{[n]}(z,\zeta) = \sum_{k=-\infty}^{L} Q^{[n]}_k(z)\zeta^k \ . \tag{6.15}$$

then eq. (6.9) implies that the two highest components of $Q^{[n]}$ are constrained as follows:

$$\begin{aligned} \mathcal{D}^{\underline{1}}_{\hat{\alpha}} Q^{[n]}_L &= 0 \ , \\ (\hat{\mathcal{D}}^{\underline{1}})^2 Q^{[n]}_{L-1} &= -12\omega\, J\, Q^{[n]}_L \ , \end{aligned} \tag{6.16}$$

Therefore, $Q^{[n]}_L$ is a five-dimensional *antichiral superfield*, while $Q^{[n]}_{L-1}$ a *complex antilinear superfield*.

For further analysis, it is useful to switch from the 5D four-component spinor notation to the 4D two-component one by representing

$$\mathcal{D}^i_{\hat{\alpha}} = \begin{pmatrix} \mathcal{D}^i_\alpha \\ \bar{\mathcal{D}}^{\dot{\alpha} i} \end{pmatrix} = \begin{pmatrix} \mathcal{D}^i_\alpha \\ \varepsilon^{ij}\bar{\mathcal{D}}^{\dot{\alpha}}_j \end{pmatrix} \ .$$

In such a notation, the algebra of covariant derivatives (2.21a–2.21c) takes the form

$$\{\mathcal{D}^i_\alpha, \mathcal{D}^j_\beta\} = 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\mathcal{D}_5 - 3\omega\varepsilon^{ij}\varepsilon_{\alpha\beta}J - 4\omega J^{ij}M_{\alpha\beta} \ , \tag{6.17a}$$

$$\{\mathcal{D}^i_\alpha, \bar{\mathcal{D}}^{\dot\beta}_j\} = -2\mathrm{i}\delta^i_j(\sigma^a)_\alpha{}^{\dot\beta}\mathcal{D}_a - 4\omega J^i{}_j M_\alpha{}^{\dot\beta} \ , \tag{6.17b}$$

$$\{\bar{\mathcal{D}}^{\dot\alpha}_i, \bar{\mathcal{D}}^{\dot\beta}_j\} = -2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\mathcal{D}_5 - 3\omega\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}J - 4\omega J_{ij}M^{\dot\alpha\dot\beta} \ , \tag{6.17c}$$

$$[\mathcal{D}_a, \mathcal{D}^i_\alpha] = -\tfrac{\mathrm{i}}{2}\omega J^{ij}(\sigma_a)_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta}_j \ , \quad [\mathcal{D}_a, \bar{\mathcal{D}}^{\dot\alpha}_i] = \tfrac{\mathrm{i}}{2}\omega J_{ij}(\tilde\sigma_a)^{\dot\alpha\beta}\mathcal{D}^j_\beta \ , \tag{6.17d}$$

$$[\mathcal{D}_5, \mathcal{D}^i_\alpha] = \tfrac{1}{2}\omega J^i{}_j\mathcal{D}^j_\alpha \ , \quad [\mathcal{D}_5, \bar{\mathcal{D}}^{\dot\alpha}_i] = \tfrac{1}{2}\omega J_i{}^j\bar{\mathcal{D}}^{\dot\beta}_j \ , \tag{6.17e}$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -\omega^2 J^2 M_{ab} \ , \quad [\mathcal{D}_a, \mathcal{D}_5] = -\omega^2 J^2 M_{a5} \ . \tag{6.17f}$$

In the two-component spinor notation, the analyticity condition $\mathcal{D}^+_{\hat\alpha}Q^{[n]}(\zeta) = 0$ is equivalent to

$$\mathcal{D}^{\underline{2}}_\alpha Q^{[n]}(\zeta) = \zeta\,\mathcal{D}^{\underline{1}}_\alpha Q^{[n]}(\zeta) \ , \qquad \bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}} Q^{[n]}(\zeta) = -\frac{1}{\zeta}\,\bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}} Q^{[n]}(\zeta) \ . \tag{6.18}$$

For the component superfields $Q^{[n]}_k(z)$, this implies

$$\mathcal{D}^{\underline{2}}_\alpha Q^{[n]}_k = \mathcal{D}^{\underline{1}}_\alpha Q^{[n]}_{k-1} \ , \qquad \bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}} Q^{[n]}_k = -\bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}} Q^{[n]}_{k+1} \ . \tag{6.19}$$

By analogy with the flat case, these constraints indicate an interesting interpretation. Let us introduce two sets of spinor derivatives, $(\mathcal{D}^{\underline{1}}_\alpha, \bar{\mathcal{D}}^{\dot\beta}_{\underline{1}})$ and $(\mathcal{D}^{\underline{2}}_\alpha, \bar{\mathcal{D}}^{\dot\beta}_{\underline{2}})$ which can be viewed as the covariant derivatives corresponding to two different sets of Grassmann variables $\Theta_{\underline{1}}$ and $\Theta_{\underline{2}}$. Then, the above constraints imply that the dependence of the projective superfields on $\Theta_{\underline{2}}$ is uniquely determined in terms of their dependence on $\Theta_{\underline{1}}$. Unlike the flat case, such an interpretation is somewhat limited in the sense that one can not consistently switch off the variables $\Theta_{\underline{2}}$ (what would be necessary for reducing the multiplets to 4D $\mathcal{N} = 1$ superfields). It follows from the algebra of covariant derivatives, specifically from eq. (6.17d), that $[\mathcal{D}_a, \mathcal{D}^{\underline{1}}_\alpha] = -\frac{\mathrm{i}}{2}\omega J^{\underline{12}}(\sigma_a)_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta}_{\underline{2}}$ and $[\mathcal{D}_a, \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}] = -\frac{\mathrm{i}}{2}\omega J^{\underline{12}}(\tilde\sigma_a)^{\dot\alpha\beta}\mathcal{D}^{\underline{2}}_\beta$, and therefore the commutation relations mix all the spinor derivatives. This is an important difference between the flat and curved cases.

The constraints (6.19) simplify if the series in (6.4) or (6.6) is bounded from below (above). Consider a real $O(2n)$ multiplet $H^{(2n)}(z, u^+)$. In accordance with the above general consideration, it can be described by the superfield $H^{[2n]}(z,\zeta)$ which is defined by $H^{(2n)}(z, u^+) = (\mathrm{i}u^{+\underline{1}}u^{+\underline{2}})^n H^{[2n]}(z,\zeta)$, and can be represented in the form

$$H^{[2n]}(z,\zeta) = \sum_{k=-n}^{+n} H^{[2n]}_k(z)\zeta^k \ , \qquad \bar{H}^{[2n]}_k = (-1)^k H^{[2n]}_{-k} \ . \tag{6.20}$$

The analyticity constraints (6.19) imply that the two lowest component superfields are constrained by

$$\begin{aligned} \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}} H^{[2n]}_{-n} &= 0 \ , \\ (\bar{\mathcal{D}}_{\underline{1}})^2 H^{[2n]}_{-n+1} &= -(4\mathcal{D}_5 + 6\,\omega J) H^{[2n]}_{-n} = -(4\mathcal{D}_5 - 12n\,\omega J^{\underline{1}}{}_{\underline{1}}) H^{[2n]}_{-n} \ , \end{aligned} \tag{6.21}$$

where we have defined

$$(\mathcal{D}^i)^2 \equiv \mathcal{D}^{i\alpha}\mathcal{D}^i_\alpha \ , \qquad (\bar{\mathcal{D}}_i)^2 \equiv \bar{\mathcal{D}}_{i\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}_i \ . \tag{6.22}$$

Consider an arctic multiplet of weight $n \geq 0$, $\Upsilon^{(n)}(u^+)$, defined to be holomorphic on $\mathbb{C}P^1 - \{N\}$. It can be represented as

$$\Upsilon^{(n)}(z, u^+) = (u^{+\underline{1}})^n\, \Upsilon^{[n]}(z,\zeta) \ , \qquad \Upsilon^{[n]}(z,\zeta) = \sum_{k=0}^{+\infty} \Upsilon^{[n]}_k(z)\zeta^k \ . \tag{6.23}$$

Then the constraints on the two lowest components superfields are

$$\begin{aligned} \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}} \Upsilon^{[n]}_0 &= 0 \ , \\ (\bar{\mathcal{D}}_{\underline{1}})^2 \Upsilon^{[n]}_1 &= -(4\mathcal{D}_5 + 6\,\omega J)\Upsilon^{[n]}_0 \ = \ -(4\mathcal{D}_5 - 6n\,\omega J^{\underline{1}}{}_{\underline{1}})\Upsilon^{[n]}_0 \ . \end{aligned} \tag{6.24}$$

In the flat superspace limit, $\omega \to 0$, the constraints (6.21) and (6.24) reduce to those given in [10].

6.2 Projective action

Here we turn to a more detailed analysis of the projective action (5.38). In the projective gauge ($u^-_{\underline{2}} = 0$, $J^{\underline{11}} = J^{\underline{22}} = 0$) used throughout this section, we have $J^{--} = 0$, and therefore the projective action simplifies

$$S = -\frac{1}{2\pi}\oint \frac{u^+_i \mathrm{d}u^{+i}}{(u^+u^-)^4}\int \mathrm{d}^5x\, e\, (\hat{\mathcal{D}}^-)^4\, \mathcal{L}^{++}\Big| \ . \tag{6.25}$$

Of course, the Lagrangian $\mathcal{L}^{++}$ should be real with respect to the smile conjugation, and can be represented as

$$\mathcal{L}^{++}(z,u^+) = \mathrm{i}u^{+\underline{1}}u^{+\underline{2}}\mathcal{L}(z,\zeta) \ . \tag{6.26}$$

Then, the action turns into

$$S = -\frac{1}{32}\oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}}\int \mathrm{d}^5x\, e\, \zeta\, (\hat{\mathcal{D}}^{\underline{1}})^2(\hat{\mathcal{D}}^{\underline{1}})^2\mathcal{L}(z,\zeta)\Big| \ , \tag{6.27}$$

where we have taken into account the fact that $\{\mathcal{D}^{\underline{1}}_{\hat\alpha}, \mathcal{D}^{\underline{1}}_{\hat\beta}\} = 0$ in the projective gauge, and also made use of the identity $\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta} = \big(\varepsilon^{\hat\alpha\hat\beta}\varepsilon^{\hat\gamma\hat\delta} + \varepsilon^{\hat\alpha\hat\gamma}\varepsilon^{\hat\delta\hat\beta} + \varepsilon^{\hat\alpha\hat\delta}\varepsilon^{\hat\beta\hat\gamma}\big)$. Using the relation

$$(\hat{\mathcal{D}}^{\underline{2}})^2 Q^{[n]} = \zeta^2 (\hat{\mathcal{D}}^{\underline{1}})^2 Q^{[n]} + 12\omega\zeta\, J Q^{[n]} \ , \tag{6.28}$$

we can express action (6.27) in the equivalent forms

$$S = -\frac{1}{32}\oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \int \mathrm{d}^5x\, e\, (\hat{\mathcal{D}}^{\underline{1}})^2 \Big((\hat{\mathcal{D}}^{\underline{2}})^2 - \zeta\, 12\omega J \Big) \mathcal{L}(z,\zeta)\Big| \ , \tag{6.29a}$$

and

$$S = -\frac{1}{32}\oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \int \mathrm{d}^5x\, e\, (\hat{\mathcal{D}}^{\underline{2}})^2 \Big((\hat{\mathcal{D}}^{\underline{1}})^2 + \frac{1}{\zeta}\, 12\omega J \Big) \mathcal{L}(z,\zeta)\Big| \ , \tag{6.29b}$$

where we have used the identities

$$[\mathcal{D}^{\underline{1}}_{\hat\alpha}, (\hat{\mathcal{D}}^{\underline{2}})^2] = -18\omega J^{\underline{1}}{}_{\underline{1}} \mathcal{D}^{\underline{2}}_{\hat\alpha} + 4\mathrm{i}\mathcal{D}_{\hat\alpha\hat\beta}\mathcal{D}^{\underline{2}\hat\beta} + 6\omega \mathcal{D}^{\underline{2}}_{\hat\alpha} J - 8\omega J^{\underline{1}}{}_{\underline{1}} \mathcal{D}^{\underline{2}\hat\beta} M_{\hat\alpha\hat\beta} \ , \tag{6.30}$$

$$[(\hat{\mathcal{D}}^{\underline{1}})^2, (\hat{\hat{\mathcal{D}}}{}^{\underline{2}})^2] = 4\mathrm{i}\mathcal{D}^{\hat\alpha\hat\beta}[\mathcal{D}^{\underline{1}}_{\hat\alpha}, \mathcal{D}^{\underline{2}}_{\hat\beta}] + (6\omega[\mathcal{D}^{\underline{1}}_{\hat\alpha}, \mathcal{D}^{\underline{2}}_{\hat\beta}] + 96\omega^2 J^{\underline{1}}{}_{\underline{1}}) J - 8\omega J^{\underline{1}}{}_{\underline{1}} [\mathcal{D}^{\underline{1}}_{\hat\alpha}, \mathcal{D}^{\underline{2}}_{\hat\beta}] M^{\hat\alpha\hat\beta} \ . \tag{6.31}$$

Then we can represent the action in the form

$$S = -\frac{1}{32}\oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \int \mathrm{d}^5x\, e \Big(\Big\{ (\hat{\mathcal{D}}^{\underline{1}})^2, (\hat{\mathcal{D}}^{\underline{2}})^2 \Big\} + 24\omega J^{\underline{1}}{}_{\underline{1}} \Big[\zeta (\hat{\mathcal{D}}^{\underline{1}})^2 + \frac{1}{\zeta} (\hat{\mathcal{D}}^{\underline{2}})^2 \Big] \Big) \mathcal{L}(z,\zeta) \Big| \tag{6.32}$$

which makes manifest the reality of S with respect to the smile-conjugation.

It can be seen from the above relations that there exists a natural "gauge freedom" in the choice of $\mathcal{L}^{++}$. It occurs in the three incarnations:

$$\mathcal{L}^{++} \ \rightarrow \ \mathcal{L}^{++} + \Lambda^{++} + \widetilde{\Lambda}^{++} \ , \tag{6.33}$$

$$\mathcal{L}^{++} \ \rightarrow \ \mathcal{L}^{++} + \mathrm{i}\, J^{++} \big(\Lambda + \widetilde{\Lambda}\big) \ , \tag{6.34}$$

$$\mathcal{L}^{++} \ \rightarrow \ \mathcal{L}^{++} + H^{++} \ , \tag{6.35}$$

with Λ^{++} and Λ arctic multiplets (6.23) of weight +2 and 0, respectively, and H^{++} a real $O(2)$ multiplet.

It is also instructive to express the action in a 4D $\mathcal{N} = 1$ form by switching to the two-component spinor notation

$$\mathcal{D}^{\underline{1}}_{\hat\alpha} = \begin{pmatrix} \mathcal{D}^{\underline{1}}_{\alpha} \\ \bar{\mathcal{D}}^{\underline{1}\dot\alpha} \end{pmatrix} = \begin{pmatrix} \mathcal{D}^{\underline{1}}_{\alpha} \\ \bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}} \end{pmatrix} \ , \qquad \mathcal{D}^{\underline{2}}_{\hat\alpha} = \begin{pmatrix} \mathcal{D}^{\underline{2}}_{\alpha} \\ \bar{\mathcal{D}}^{\underline{2}\dot\alpha} \end{pmatrix} = \begin{pmatrix} \mathcal{D}^{\underline{2}}_{\alpha} \\ -\bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}} \end{pmatrix} \ . \tag{6.36}$$

Using the analyticity conditions (6.18) we can express $\bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}}$ via $\bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}$. As a result our action (6.25) becomes

$$S = \oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \int \mathrm{d}^5x\, e \left(\frac{1}{16} (\mathcal{D}^{\underline{1}})^2 (\bar{\mathcal{D}}_{\underline{1}})^2 - \zeta\omega J^{\underline{1}}{}_{\underline{1}} (\mathcal{D}^{\underline{1}})^2 \right) \mathcal{L}(z,\zeta) \Big| \ . \tag{6.37}$$

Using the identities

$$\begin{aligned}
[\bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}, (\mathcal{D}^{\underline{1}})^2] &= 8\omega J^{\underline{1}}{}_{\underline{1}} \bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}} - 4\mathrm{i}(\tilde\sigma^a)^{\dot\alpha\alpha} \mathcal{D}_a \mathcal{D}^{\underline{1}}_\alpha + 8\omega J^{\underline{1}}{}_{\underline{1}} \mathcal{D}^{\underline{1}\alpha} M_\alpha{}^{\dot\alpha} \ , &(6.38)\\
[(\mathcal{D}^{\underline{1}})^2, (\bar{\mathcal{D}}_{\underline{1}})^2] &= -16\omega J^{\underline{1}}{}_{\underline{1}} \bar{\mathcal{D}}_{\underline{1}\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}} + 16\omega J^{\underline{1}}{}_{\underline{1}} \mathcal{D}^{\underline{1}\alpha} \mathcal{D}^{\underline{2}}_\alpha + 96\omega^2 J^{\underline{1}}{}_{\underline{1}} J \\
&\quad + 4\mathrm{i}(\sigma^a)_\alpha{}^{\dot\alpha} \mathcal{D}_a [\mathcal{D}^{\underline{1}\alpha}, \bar{\mathcal{D}}_{\underline{1}\dot\alpha}] + 8\omega J^{\underline{1}}{}_{\underline{1}} [\mathcal{D}^{\underline{1}\alpha}, \bar{\mathcal{D}}_{\underline{1}\dot\alpha}] M_\alpha{}^{\dot\alpha} \ , &(6.39)
\end{aligned}$$

the action can also be rewritten in the following form

$$S = \oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \int \mathrm{d}^5x\, e \left(\frac{1}{16} (\bar{\mathcal{D}}_{\underline{1}})^2 (\mathcal{D}^{\underline{1}})^2 + \frac{\omega}{\zeta} J^{\underline{1}}{}_{\underline{1}} (\bar{\mathcal{D}}_{\underline{1}})^2 \right) \mathcal{L}(z,\zeta) \Big| \ , \tag{6.40}$$

or in the manifestly real form

$$S = \oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \int \mathrm{d}^5x\, e \left(\frac{1}{32} \Big\{ (\mathcal{D}^{\underline{1}})^2, (\bar{\mathcal{D}}_{\underline{1}})^2 \Big\} - \frac{1}{2}\zeta\omega J^{\underline{1}}{}_{\underline{1}} (\mathcal{D}^{\underline{1}})^2 + \frac{\omega}{2\zeta} J^{\underline{1}}{}_{\underline{1}} (\bar{\mathcal{D}}_{\underline{1}})^2 \right) \mathcal{L}(z,\zeta) \Big| \ . \tag{6.41}$$

As compared with the flat superspace action [10], the second and third terms on the right of (6.41) are due to the non-vanishing curvature.

6.3 Nonlinear sigma-models

We consider a system of interacting artic weight-one multiplets $\Upsilon^+(z,u^+)$ and their smile-conjugates $\widetilde{\Upsilon}^+$ described by the Lagrangian

$$\mathcal{L}^{++} = \mathrm{i}\, K(\Upsilon^+, \widetilde{\Upsilon}^+) \ , \tag{6.42}$$

with $K(\Phi^I, \bar\Phi^{\bar J})$ a real analytic function. Since $\mathcal{L}^{++} = \mathcal{L}^{++}(z,u^+)$ is required to be a weight-two projective superfield, the potential K has to respect the following homogeneity condition

$$\Big(\Phi^I \frac{\partial}{\partial \Phi^I} + \bar\Phi^{\bar I} \frac{\partial}{\partial \bar\Phi^{\bar I}} \Big) K(\Phi, \bar\Phi) = 2\, K(\Phi, \bar\Phi) \ . \tag{6.43}$$

For $\mathcal{L}^{++}$ to be real, it is sufficient to require a stronger condition

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar\Phi) = K(\Phi, \bar\Phi) \ . \tag{6.44}$$

Such a Lagrangian corresponds to the superconformal sigma-model introduced in [25]. Then, representing $\Upsilon^+(z,u^+) = u^{+\underline{1}}\,\Upsilon(z,\zeta)$ and $\widetilde{\Upsilon}^+(z,u^+) = u^{+\underline{2}}\,\widetilde{\Upsilon}(z,\zeta)$, we can rewrite the Lagrangian in the form

$$\mathcal{L}^{++}(z,u^+) = {\rm i}\,u^{+\underline{1}}u^{+\underline{2}}\,\mathcal{L}(z,\zeta)~, \qquad \mathcal{L} = K(\Upsilon,\widetilde{\Upsilon})~. \tag{6.45}$$

Because of freedom (6.33) in the choice of Lagrangian, we can generalize the above construction by replacing $K(\Phi^I,\bar{\Phi}^{\bar{J}})$ in (6.42) with

$$K'(\Phi^I,\bar{\Phi}^{\bar{J}}) = K(\Phi^I,\bar{\Phi}^{\bar{J}}) + \Lambda(\Phi^I) - \bar{\Lambda}(\bar{\Phi}^{\bar{J}})~, \qquad \Phi^I\frac{\partial}{\partial\Phi^I}\Lambda(\Phi) = 2\,\Lambda(\Phi)~, \tag{6.46}$$

with $\Lambda(\Phi)$ a holomorphic homogeneous function of degree $+2$. Then, the homogeneity condition (6.44) turns into

$$\Phi^I\frac{\partial}{\partial\Phi^I}K'(\Phi,\bar{\Phi}) = K'(\Phi,\bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi})~. \tag{6.47}$$

We can also consider a system of interacting arctic weight-zero multiplets $\boldsymbol{\Upsilon}(z,u^+)$ and their smile-conjugates $\widetilde{\boldsymbol{\Upsilon}}$ described by the Lagrangian

$$\mathcal{L}^{++} = \frac{\rm i}{2}\,J^{++}\,\mathbf{K}(\boldsymbol{\Upsilon},\widetilde{\boldsymbol{\Upsilon}})~, \tag{6.48}$$

with $\mathbf{K}(\Phi^I,\bar{\Phi}^{\bar{J}})$ a real function which is not required to obey any homogeneity condition. Due to the gauge freedom (6.34), the action is invariant under Kähler transformations of the form

$$\mathbf{K}(\boldsymbol{\Upsilon},\widetilde{\boldsymbol{\Upsilon}}) ~\to~ \mathbf{K}(\boldsymbol{\Upsilon},\widetilde{\boldsymbol{\Upsilon}}) + \boldsymbol{\Lambda}(\boldsymbol{\Upsilon}) + \bar{\boldsymbol{\Lambda}}(\widetilde{\boldsymbol{\Upsilon}})~, \tag{6.49}$$

with $\boldsymbol{\Lambda}$ a holomorphic function. Such dynamical systems generalize the hyperkähler sigma-models on cotangent bundles of Kähler manifolds [39, 40, 41].

6.4 Vector multiplet and Chern-Simons couplings

An Abelian vector mulitplet can be described by a weight-zero real projective superfield $V(z,u^+)$ which is required to be holomorphic on $\mathbb{C}P^1 - \{N \cup S\}$.

$$\mathcal{D}^+_{\hat{\alpha}}V(z,u^+) = 0~, \qquad V(z,c\,u^+) = V(z,u^+)~, \qquad c \in \mathbb{C}^*~. \tag{6.50}$$

In the North chart, it is characterized by the series (5.22). It is defined to possess the gauge freedom

$$V ~\to~ V + \lambda + \widetilde{\lambda}~, \qquad \lambda(z,\zeta) = \sum_{k=0}^{+\infty}\lambda_k(z)\zeta^k~. \tag{6.51}$$

with $\lambda(z,u^+)$ an arctic multiplet of weight 0. Using considerations similar to those given in subsection 5.2, the field strength (compare with the flat superspace expression [25])

$$W(z) = -\frac{1}{16\pi \mathrm{i}} \oint \frac{u_i^+ \,\mathrm{d}u^{+i}}{(u^+u^-)^2} \Big[(\hat{\mathcal{D}}^-)^2 - 12\omega\, J^{--}\Big] V(z,u^+) \tag{6.52}$$

can be shown to be invariant under the projective transformations (5.3). The field strength turns out to be invariant under the gauge transformations (6.51). In the projective gauge ($u^-_{\underline{2}} = 0$, $J^{\underline{11}} = J^{\underline{22}} = 0$), the field strength takes the form

$$W(z) = -\frac{1}{16\pi \mathrm{i}} \oint \mathrm{d}\zeta\, (\hat{\mathcal{D}}^{\underline{1}})^2 V(z,\zeta) \ , \tag{6.53}$$

compare with the flat superspace result [10].

The AdS transformation law of V,

$$\delta V = -\big(\xi + \mathrm{i}\,\rho J\big) V \ , \tag{6.54}$$

can be shown to imply that W transforms as

$$\delta W = -\xi\, W \tag{6.55}$$

under the isometry group.

The field strength can be shown to obey the Bianchi identity

$$\mathcal{D}^{(i}_{\hat{\alpha}} \mathcal{D}^{j)}_{\hat{\beta}} W = \frac{1}{4} \varepsilon_{\hat{\alpha}\hat{\beta}}\, \mathcal{D}^{\hat{\gamma}(i} \mathcal{D}^{j)}_{\hat{\gamma}} W \ , \tag{6.56}$$

and therefore

$$\mathcal{D}^{(i}_{\hat{\alpha}} \mathcal{D}^{j}_{\hat{\beta}} \mathcal{D}^{k)}_{\hat{\gamma}} \mathcal{W} = 2\omega\, \varepsilon_{\hat{\beta}\hat{\gamma}}\, J^{(ij} \mathcal{D}^{k)}_{\hat{\alpha}} W \ , \tag{6.57}$$

compare with the flat superspace case [9, 10]. The Bianchi identity implies that

$$G^{++}(z,u^+) = \mathrm{i}\Big\{\mathcal{D}^{+\hat{\alpha}} W\, \mathcal{D}^+_{\hat{\alpha}} W + \frac{1}{2} W\, (\hat{\mathcal{D}}^+)^2 W - 2\omega\, J^{++} W^2\Big\} \tag{6.58}$$

is a composite $O(2)$ multiplet,

$$\mathcal{D}^+_{\hat{\alpha}} G^{++} = 0 \ , \qquad G^{++}(z,u^+) = G^{ij}(z)\, u^+_i u^+_j \ . \tag{6.59}$$

Let $H^{++}(z,u^+)$ be a real $O(2)$ multiplet. Then, similarly to the flat superspace case [10, 25], the supersymmetric action associated with the Lagrangian

$$\mathcal{L}^{++} = V(z,u^+)\, H^{++}(z,u^+) \tag{6.60}$$

can be shown to be invariant under the gauge transformations (6.51).

Given several Abelian vector multiplets $V_I(z,u^+)$, where $I = 1,\dots,n$, the composite superfield (6.58) is generalised to the form:

$$G^{++}_{IJ} = G^{++}_{(IJ)} = \mathrm{i}\Big\{\mathcal{D}^{+\hat\alpha}W_I\,\mathcal{D}^+_{\hat\alpha}W_J + \frac{1}{2}\,W_{(I}\,(\hat{\mathcal{D}}^+)^2W_{J)} - 2\omega\,J^{++}W_IW_J\Big\}\ ,$$
$$\mathcal{D}^+_{\hat\alpha}G^{++}_{IJ} = 0\ , \qquad G^{++}_{IJ}(z,u^+) = G^{ij}_{IJ}(z)\,u^+_iu^+_j\ . \tag{6.61}$$

We then can construct a supersymmetric Chern-Simons action associated with the Lagrangian

$$\mathcal{L}^{++}_{\rm CS} = \frac{1}{12}\,c_{I,JK}\,V_I(z,u^+)\,G^{++}_{JK}(z,u^+)\ , \qquad c_{I,JK} = c_{I,KJ}\ , \tag{6.62}$$

for some constant parameters $c_{I,JK}$ (compare with the flat superspace case [10, 25]). In accordance with the above result, the Chern-Simons action is gauge invariant.

6.5 Tensor multiplet and vector-tensor couplings

Given several $O(2)$ (or, equivalently, tensor) multiplets $H^{++}_I(z,u^+)$, a supersymmetric action is generated by the Lagrangian

$$\mathcal{L}^{++} = \mathcal{F}(H^{++}_I)\ , \qquad I = 1,\dots,n \tag{6.63}$$

where $\mathcal{F}(H)$ is a weakly homogeneous function of first degree in the variables H,

$$H_I\,\frac{\partial\mathcal{F}(H)}{\partial H_I} - \mathcal{F}(H) = \alpha^I\,H_I\ , \tag{6.64}$$

for some constants α's.[8] Such a Lagrangian occurs in the models for superconformal tensor multiplets in four [42] and five dimensions [25].

One can also consider systems of coupled vector and tensor multiplets described by a Lagrangian of the form

$$\mathcal{L}^{++} = \mathcal{F}(H^{++}_I) + V_I\Big(\kappa_IH^{++}_I + \frac{1}{12}\,c_{I,JK}\,G^{++}_{JK}\Big)\ , \tag{6.65}$$

for some coupling constants κ_I and $c_{I,JK}$.

[8]The projective action principle formulated in subsection 5.2 requires the Lagrangian to be a projective weight-two multiplet. With $\alpha^I \neq 0$ in (6.64), the Lagrangian (6.63) does not have any definite weight, and hence the results of subsection 5.2 are not applicable directly. We plan to discuss the case with $\alpha^I \neq 0$ in more detail somewhere else.

7 Coset space realization

In this section we would like to give an explicit realization for the $\mathcal{N}=1$ AdS$_5$ supergeometry which we have studied in section 2 using the representation-independent approach. From the group-theoretical point of view, it is known that the $\mathcal{N}=1$ AdS$_5$ superspace (or simply AdS$^{5|8}$) can be identified with the coset space SU(2,2|1)/SO(4,1)×U(1). Using the formalism of nonlinear realizations[9] [44] (or Cartan's coset construction), here we introduce a suitable coset representative that makes possible to realize one half of AdS$^{5|8}$ as a trivial fiber bundle with fibers isomorophic to four-dimensional Minskowski superspace. This realization should be useful if one is interested in having the 4D $\mathcal{N}=1$ super Poincaré symmetry manifest. However, since it corresponds to one half of AdS$^{5|8}$ (known as the Poincaré patch [45]), it is not suitable to describe the supersymmetric actions.

The analysis of this section builds on the construction given in [46], see also [47] for related issues. Note that we use the superform conventions of [19].

7.1 Coset representative

As is well known, the supergroup SU(2,2|1) is the four-dimensional $\mathcal{N}=1$ superconformal group. It is generated by Lie-algebra elements of the form (parametrization (7.1) was used in [48, 49])

$$X = \begin{pmatrix} w_\alpha{}^\beta - \Delta\delta_\alpha{}^\beta & -\mathrm{i}b_{\alpha\dot\beta} & 2\rho_\alpha \\ -\mathrm{i}a^{\dot\alpha\beta} & -\bar w^{\dot\alpha}{}_{\dot\beta} + \bar\Delta\delta^{\dot\alpha}{}_{\dot\beta} & 2\bar\epsilon^{\dot\alpha} \\ 2\epsilon^\beta & 2\bar\rho_{\dot\beta} & 2(\bar\Delta - \Delta) \end{pmatrix} , \tag{7.1}$$

which satisfy the conditions

$$\mathrm{str}\, X = 0 \ , \qquad BX^\dagger B = -X \ , \qquad B = \begin{pmatrix} 0 & \mathbb{1} & 0 \\ \mathbb{1} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{7.2}$$

The matrix elements in (7.1) correspond to a 4D Lorentz transformation $(w_\alpha{}^\beta,\ \bar w^{\dot\alpha}{}_{\dot\beta})$, a translation $a^{\dot\alpha\alpha}$, a special conformal transformation $b_{\alpha\dot\alpha}$, a Q–supersymmetry $(\epsilon^\alpha,\ \bar\epsilon^{\dot\alpha})$, an S–supersymmetry $(\rho_\alpha,\ \bar\rho_{\dot\alpha})$, and a combined scale and U(1)–chiral transformation $\Delta = \frac{1}{2}\lambda + \frac{\mathrm{i}}{3}\tau$.

[9]Many years ago, this formalism was also applied to introduce the 4D $\mathcal{N}=1$ AdS superspace [43, 27].

The explicit parametrization for the algebra su$(2,2|1)$, which is given in (7.1), is ideally suited to describe the compactified Minkowski space SU(2,2|1)/$(\mathcal{P}\times\mathbb{C}^*)$, where $\mathcal{P}$ denotes the $\mathcal{N}=1$ super Poincaré group (generated by the parameters $(w, \bar{w},\ b,\ \rho,\ \bar{\rho})$ in (7.1)), and $\mathbb{C}^*$ denotes the group of scale and chiral transformations generated by the parameters Δ and $\bar{\Delta}$ in (7.1). In the case of the coset space SU(2,2|1)/SO(4,1)×U(1), however, this parametrization should be slightly modified. In addition, a re-scaling of some matrix elements is needed in order to incorporate the AdS curvature ω^2 into the formalism.

As is known, a key role in the coset construction for $\mathcal{M}=G/H$ is played by a coset representative $S(p)$ defined to be a smooth mapping S: $U\rightarrow G$, for some open domain $U\subset\mathcal{M}$, such that $S(p)p_0=p$, for any point $p\in U$, where $p_0\in U$ is a fixed point having H as its isotropy group. On topological grounds, it is not always possible to extend U to the whole coset space $\mathcal{M}$.

As a coset representative, $S(z)$, for AdS$^{5|8}$ = SU(2,2|1)/SO(4,1)×U(1), following mainly [46] we choose

$$\begin{aligned} S(z) &= g(\boldsymbol{z})\cdot g_S\cdot g_D \\ &= \begin{pmatrix} \mathbb{1} & 0 & 0 \\ -\mathrm{i}\omega\tilde{x}_+ & \mathbb{1} & 2\omega^{\frac12}\bar{\theta} \\ 2\omega^{\frac12}\theta & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 2\omega\eta\bar{\eta} & 2\omega^{\frac12}\eta \\ 0 & \mathbb{1} & 0 \\ 0 & 2\omega^{\frac12}\bar{\eta} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\frac12\omega y}\mathbb{1} & 0 & 0 \\ 0 & \mathrm{e}^{\frac12\omega y}\mathbb{1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{-\frac12\omega y}\delta_\alpha{}^\beta & 2\omega\mathrm{e}^{\frac12\omega y}\eta_\alpha\bar{\eta}_{\dot\beta} & 2\omega^{\frac12}\eta_\alpha \\ -\mathrm{i}\omega\mathrm{e}^{-\frac12\omega y}\tilde{x}_+^{\dot\alpha\beta} & \mathrm{e}^{\frac12\omega y}\big(\delta^{\dot\alpha}{}_{\dot\beta}-2\mathrm{i}\omega^2\tilde{x}_+^{\dot\alpha\gamma}\eta_\gamma\bar{\eta}_{\dot\beta}+4\omega\bar{\theta}^{\dot\alpha}\bar{\eta}_{\dot\beta}\big) & 2\omega^{\frac12}\big(\bar{\theta}^{\dot\alpha}-\mathrm{i}\omega\tilde{x}_+^{\dot\alpha\gamma}\eta_\gamma\big) \\ 2\omega^{\frac12}\mathrm{e}^{-\frac12\omega y}\theta^\beta & 2\omega^{\frac12}\mathrm{e}^{\frac12\omega y}\big(\bar{\eta}_{\dot\beta}+2\omega\theta^\gamma\eta_\gamma\bar{\eta}_{\dot\beta}\big) & \big(1+4\omega\theta^\gamma\eta_\gamma\big) \end{pmatrix}, \end{aligned} \qquad (7.3)$$

where $x^a_\pm=x^a\pm\mathrm{i}\theta\sigma^a\bar{\theta}$ denote ordinary 4D $\mathcal{N}=1$ (anti) chiral bosonic variables. It is worth pointing out that the coset representative $g(\boldsymbol{z})$ corresponds to the coset $\mathcal{P}$/SO(3,1) and provides a matrix realization[10] for 4D $\mathcal{N}=1$ Minkowski superspace, with coordinates $\boldsymbol{z}=(x^a,\theta^\alpha,\bar{\theta}_{\dot\alpha})$. Note that the isotropy group at $z=0$ is $H=\mathrm{SO}(4,1)\times\mathrm{U}(1)\in\mathrm{SU}(2,2|1)=G$, and it is generated by matrices of the form

$$\mathbf{H}=\begin{pmatrix} \mathbf{w} & -\frac{\mathrm{i}}{2}\mathbf{b} & 0 \\ \frac{\mathrm{i}}{2}\tilde{\mathbf{b}} & -\bar{\mathbf{w}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{\mathrm{i}}{3}\tau\,\mathbb{1} & 0 & 0 \\ 0 & -\frac{\mathrm{i}}{3}\tau\,\mathbb{1} & 0 \\ 0 & 0 & -\frac{4\mathrm{i}}{3}\tau \end{pmatrix}, \qquad \begin{array}{ll} \mathrm{tr}\,\mathbf{w}=0\ , & \bar{\mathbf{w}}=\mathbf{w}^\dagger\ , \\ \mathbf{b}^\dagger=\mathbf{b}\ , & \bar{\tau}=\tau\ . \end{array} \qquad (7.4)$$

Setting $\omega=1$ in (7.3) gives the parametrization used in [46].

[10]It is a curious historic fact that the above matirx realization for 4D $\mathcal{N}=1$ Minkowski superspace was introduced by Akulov and Volkov [50] a year before the official discovery of superspace.

Once the coset representative $S(p)$ is chosen, the next step in the coset construction for $\mathcal{M} = G/H$ is to compute the Maurer-Cartan one-form $S^{-1}\mathrm{d}S$ which proves to encode all the information about the geometry of $\mathcal{M}$. Let $\mathcal{G}$ and $\mathcal{H}$ be the Lie algebras of G and H, respectively, and $\mathcal{G}-\mathcal{H}$ be a complement of $\mathcal{H}$ in $\mathcal{G}$ such that $[\mathcal{G}-\mathcal{H}, \mathcal{H}] \subset \mathcal{G}-\mathcal{H}$. Then, the Maurer-Cartan one-form can be uniquely decomposed as $S^{-1}\mathrm{d}S = S^{-1}\mathrm{d}S|_{\mathcal{G}-\mathcal{H}} + S^{-1}\mathrm{d}S|_{\mathcal{H}}$, where $S^{-1}\mathrm{d}S|_{\mathcal{G}-\mathcal{H}}$ is identified with the vielbein, and $S^{-1}\mathrm{d}S|_{\mathcal{H}}$ with the connection.

In our case, the vielbein $\mathbf{E} = S^{-1}\mathrm{d}S|_{\mathcal{G}-\mathcal{H}}$ and the connection $\boldsymbol{\Omega} = S^{-1}\mathrm{d}S|_{\mathcal{H}}$ are:

$$
\begin{aligned}
S^{-1}\,\mathrm{d}S &= \mathbf{E} + \boldsymbol{\Omega}\ , \\
\mathbf{E} &= \begin{pmatrix} -\frac{1}{2}\omega\mathbf{E}_y\,\delta_\alpha{}^\beta & -\frac{\mathrm{i}}{2}\omega\mathbf{E}_{\alpha\dot\beta} & 2\omega^{\frac{1}{2}}(\mathbf{E}_\eta)_\alpha \\ -\frac{\mathrm{i}}{2}\omega\tilde{\mathbf{E}}^{\dot\alpha\beta} & \frac{1}{2}\omega\mathbf{E}_y\,\delta^{\dot\alpha}{}_{\dot\beta} & 2\omega^{\frac{1}{2}}(\bar{\mathbf{E}}_{\bar\theta})^{\dot\alpha} \\ 2\omega^{\frac{1}{2}}(\mathbf{E}_\theta)^\beta & 2\omega^{\frac{1}{2}}(\bar{\mathbf{E}}_{\bar\eta})_{\dot\beta} & 0 \end{pmatrix}\ , && (7.5) \\
\boldsymbol{\Omega} &= \begin{pmatrix} \boldsymbol{\Omega}_\alpha{}^\beta - \frac{\mathrm{i}}{3}\,\boldsymbol{\Omega}_{\mathrm{U}(1)}\delta_\alpha{}^\beta & -\frac{\mathrm{i}}{2}\boldsymbol{\Omega}_{\alpha\dot\beta} & 0 \\ \frac{\mathrm{i}}{2}\tilde{\boldsymbol{\Omega}}^{\dot\alpha\beta} & -\bar{\boldsymbol{\Omega}}^{\dot\alpha}{}_{\dot\beta} - \frac{\mathrm{i}}{3}\,\boldsymbol{\Omega}_{\mathrm{U}(1)}\delta^{\dot\alpha}{}_{\dot\beta} & 0 \\ 0 & 0 & -\frac{4\mathrm{i}}{3}\,\boldsymbol{\Omega}_{\mathrm{U}(1)} \end{pmatrix}\ . && (7.6)
\end{aligned}
$$

The components of the vielbein are given by the one-forms

$$
\begin{aligned}
\mathbf{E}_{\alpha\dot\alpha} &= e_{\alpha\dot\alpha}\,\mathrm{e}^{-\omega y}(1 - \mathrm{e}^{2\omega y}\omega^2\eta^2\bar\eta^2) + 2\mathrm{i}\mathrm{e}^{\omega y}\mathrm{d}\eta_\alpha\bar\eta_{\dot\alpha} + 2\mathrm{i}\mathrm{e}^{\omega y}\mathrm{d}\bar\eta_{\dot\alpha}\eta_\alpha \\
&\quad + 4\mathrm{i}\omega\mathrm{e}^{\omega y}\mathrm{d}\theta_\alpha\eta^2\bar\eta_{\dot\alpha} + 4\mathrm{i}\omega\mathrm{e}^{\omega y}\mathrm{d}\bar\theta_{\dot\alpha}\eta_\alpha\bar\eta^2\ , && (7.7a) \\
\mathbf{E}_y &= \mathrm{d}y + \mathrm{d}\theta^\mu(-2\eta_\mu) + \mathrm{d}\bar\theta_{\dot\mu}(-2\bar\eta^{\dot\mu})\ , && (7.7b) \\
(\mathbf{E}_\theta)^\alpha &= \mathrm{d}\theta^\mu\,\delta_\mu^\alpha\mathrm{e}^{-\frac{1}{2}\omega y} + e^m(\mathrm{i}\omega\mathrm{e}^{-\frac{1}{2}\omega y}\bar\eta_{\dot\beta}\tilde\sigma_m^{\dot\beta\alpha})\ , && (7.7c) \\
(\mathbf{E}_\eta)^\alpha &= \mathrm{d}\eta^\mu\,\delta_\mu^\alpha\mathrm{e}^{\frac{1}{2}\omega y} + \mathrm{d}\theta^\mu\,\delta_\mu^\alpha(2\omega\mathrm{e}^{\frac{1}{2}\omega y}\eta^2) + \mathrm{d}\bar\theta_{\dot\mu}(2\omega\mathrm{e}^{\frac{1}{2}\omega y}\bar\eta^{\dot\mu}\eta^\alpha) + e^m(\mathrm{i}\omega^2\mathrm{e}^{\frac{1}{2}\omega y}\eta^2\bar\eta_{\dot\beta}\tilde\sigma_m^{\dot\beta\alpha})\ . && (7.7d)
\end{aligned}
$$

The components of the SO(4,1)×U(1) connection read

$$
\begin{aligned}
\boldsymbol{\Omega}_\alpha{}^\beta &= \mathrm{d}\theta^\mu(4\omega\eta_\alpha\delta_\mu^\beta - 2\omega\eta_\mu\delta_\alpha^\beta) + e^m(-2\mathrm{i}\omega^2\eta_\alpha\bar\eta_{\dot\beta}\tilde\sigma_m^{\dot\beta\beta} - \mathrm{i}\omega^2\bar\eta_{\dot\gamma}\tilde\sigma_m^{\dot\gamma\gamma}\eta_\gamma\delta_\alpha^\beta)\ , && (7.8a) \\
\boldsymbol{\Omega}_{\alpha\dot\alpha} &= -e_{\alpha\dot\alpha}\,\omega\mathrm{e}^{-\omega y}(1 + \omega^2\mathrm{e}^{2\omega y}\eta^2\bar\eta^2) + \mathrm{d}\eta_\alpha(2\mathrm{i}\omega\mathrm{e}^{\omega y}\bar\eta_{\dot\alpha}) + \mathrm{d}\bar\eta_{\dot\alpha}(2\mathrm{i}\omega\mathrm{e}^{\omega y}\eta_\alpha) \\
&\quad + \mathrm{d}\theta_\alpha(4\mathrm{i}\omega^2\mathrm{e}^{\omega y}\eta^2\bar\eta_{\dot\alpha}) + \mathrm{d}\bar\theta_{\dot\alpha}(4\mathrm{i}\omega^2\mathrm{e}^{\omega y}\eta_\alpha\bar\eta^2)\ , && (7.8b) \\
\boldsymbol{\Omega}_{\mathrm{U}(1)} &= \mathrm{d}\theta^\mu(\,3\mathrm{i}\omega\eta_\mu) + \mathrm{d}\bar\theta_{\dot\mu}(-3\mathrm{i}\omega\bar\eta^{\dot\mu}) + e^m(-3\omega^2\bar\eta_{\dot\mu}\tilde\sigma_m^{\dot\mu\mu}\eta_\mu)\ , && (7.8c)
\end{aligned}
$$

where

$$
e^m = \mathrm{d}x^m - \mathrm{i}\,\mathrm{d}\theta^\mu\sigma^a_{\mu\dot\mu}\bar\theta^{\dot\mu} + \mathrm{i}\,\theta^\mu\sigma^a_{\mu\dot\mu}\mathrm{d}\bar\theta^{\dot\mu}\ , \tag{7.9}
$$

is the space-time component of the $\mathcal{N} = 1$ flat superspace vielbein [19].

Note that under a group transformation $g \in \mathrm{SU}(2,2|1)$

$$
g\,S(z) \;=\; S(g\cdot z)\,\hat h(z;g) \;\equiv\; S'\,\hat h\ , \qquad \hat h(z;g) \in H\ , \tag{7.10}
$$

the vielbein $\mathbf{E}$ and the connection $\mathbf{\Omega}$ transform as follows:

$$\mathbf{E}' = \hat{h}\,\mathbf{E}\,\hat{h}^{-1} , \qquad \mathbf{\Omega}' = \hat{h}\,\mathbf{\Omega}\,\hat{h}^{-1} - (\mathrm{d}\hat{h})\,\hat{h}^{-1} . \tag{7.11}$$

It is useful to introduce the inverse $E_A{}^M$ of the vielbein supermatrix $E_M{}^A$ implicitly used in the previous equations ($E_A{}^M E_M{}^B = \delta_A{}^B$, $E_M{}^A E_A{}^N = \delta_M{}^N$). With the definitions

$$\varepsilon^M = \big(e^m, \mathrm{d}y, \mathrm{d}\theta^\mu, \mathrm{d}\bar{\theta}_{\dot{\mu}}, \mathrm{d}\eta^\mu, \mathrm{d}\bar{\eta}_{\dot{\mu}}\big) = \mathbf{E}^A E_A{}^M , \tag{7.12}$$

$$\mathbf{E}^A = \big(\mathbf{E}^a, \mathbf{E}_y, (\mathbf{E}_\theta)^\alpha, (\bar{\mathbf{E}}_{\bar{\theta}})_{\dot{\alpha}}, (\mathbf{E}_\eta)^\alpha, (\bar{\mathbf{E}}_{\bar{\eta}})_{\dot{\alpha}}\big) = \varepsilon^M E_M{}^A , \tag{7.13}$$

where $e^m = -\frac{1}{2}(\tilde{\sigma}^m)^{\dot{\alpha}\alpha} e_{\alpha\dot{\alpha}}$ and $\mathbf{E}^a = -\frac{1}{2}(\tilde{\sigma}^a)^{\dot{\alpha}\alpha}\mathbf{E}_{\alpha\dot{\alpha}}$, we find

$$\begin{aligned} e_{\alpha\dot{\alpha}} &= \mathbf{E}_{\alpha\dot{\alpha}}\,\mathrm{e}^{\omega y}(1+\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2) + (\mathbf{E}_\eta)_\alpha(-2\mathrm{i}\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}_{\dot{\alpha}}) + (\bar{\mathbf{E}}_{\bar{\eta}})_{\dot{\alpha}}(-2\mathrm{i}\mathrm{e}^{\frac{3}{2}\omega y}\eta_\alpha) \\ &\quad + (\mathbf{E}_\theta)_\alpha(2\mathrm{i}\omega\mathrm{e}^{\frac{5}{2}\omega y}\bar{\eta}_{\dot{\alpha}}\eta^2) + (\bar{\mathbf{E}}_{\bar{\theta}})_{\dot{\alpha}}(2\mathrm{i}\omega\mathrm{e}^{\frac{5}{2}\omega y}\eta_\alpha\bar{\eta}^2) , \end{aligned} \tag{7.14a}$$

$$\mathrm{d}y = \mathbf{E}_y + (\mathbf{E}_\theta)^\alpha(\,2\mathrm{e}^{\frac{1}{2}\omega y}\eta_\alpha) + (\bar{\mathbf{E}}_{\bar{\theta}})_{\dot{\alpha}}(\,2\mathrm{e}^{\frac{1}{2}\omega y}\bar{\eta}^{\dot{\alpha}}) , \tag{7.14b}$$

$$\begin{aligned} \mathrm{d}\theta^\mu &= (\mathbf{E}_\theta)^\alpha\,\mathrm{e}^{\frac{1}{2}\omega y}\delta^\mu_\alpha(1-2\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2) + (\mathbf{E}_\eta)^\alpha\,\delta^\mu_\alpha(2\omega\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}^2) \\ &\quad + (\bar{\mathbf{E}}_{\bar{\eta}})_{\dot{\alpha}}(-2\omega\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}^{\dot{\alpha}}\eta^\mu) + \mathbf{E}^a(-\mathrm{i}\omega\mathrm{e}^{\omega y}\bar{\eta}_{\dot{\nu}}\tilde{\sigma}_a^{\dot{\nu}\mu}) , \end{aligned} \tag{7.14c}$$

$$\begin{aligned} \mathrm{d}\eta^\mu &= (\mathbf{E}_\eta)^\alpha\,\mathrm{e}^{-\frac{1}{2}\omega y}\delta^\mu_\alpha(1-4\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2) + (\mathbf{E}_\theta)^\alpha\,\delta^\mu_\alpha(-2\omega\mathrm{e}^{\frac{1}{2}\omega y}\eta^2) \\ &\quad + (\bar{\mathbf{E}}_{\bar{\theta}})_{\dot{\alpha}}(-2\omega\mathrm{e}^{\frac{1}{2}\omega y}\bar{\eta}^{\dot{\alpha}}\eta^\mu) + \mathbf{E}^a(2\mathrm{i}\omega^2\mathrm{e}^{\omega y}\bar{\eta}_{\dot{\nu}}\tilde{\sigma}_a^{\dot{\nu}\mu}\eta^2) . \end{aligned} \tag{7.14d}$$

It is also useful to decompose the connection with respect to the curved basis $\{\mathbf{E}^A\}$

$$\begin{aligned} \mathbf{\Omega}_\alpha{}^\beta &= (\mathbf{E}_\theta)^\gamma\,\omega\mathrm{e}^{\frac{1}{2}\omega y}(4\eta_\alpha\delta^\beta_\gamma - 2\eta_\gamma\delta^\beta_\alpha) + (\mathbf{E}_\eta)^\gamma\,\omega\mathrm{e}^{\frac{3}{2}\omega y}(4\eta_\alpha\bar{\eta}^2\delta^\beta_\gamma - 2\eta_\gamma\bar{\eta}^2\delta^\beta_\alpha) \\ &\quad + \mathbf{E}^a\,\omega^2\mathrm{e}^{\omega y}(-2\mathrm{i}\bar{\eta}_{\dot{\beta}}\tilde{\sigma}_a^{\dot{\beta}\beta}\eta_\alpha + \mathrm{i}\bar{\eta}_{\dot{\gamma}}\tilde{\sigma}_a^{\dot{\gamma}\gamma}\eta_\gamma\delta^\beta_\alpha) , \end{aligned} \tag{7.15a}$$

$$\begin{aligned} \mathbf{\Omega}_{\alpha\dot{\alpha}} &= \mathbf{E}_{\alpha\dot{\alpha}}\,\omega(-1-2\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2) + (\mathbf{E}_\eta)_\alpha(4\mathrm{i}\omega\mathrm{e}^{\frac{1}{2}\omega y}\bar{\eta}_{\dot{\alpha}}) + (\bar{\mathbf{E}}_{\bar{\eta}})_{\dot{\alpha}}(4\mathrm{i}\omega\mathrm{e}^{\frac{1}{2}\omega y}\eta_\alpha) \\ &\quad + (\mathbf{E}_\theta)_\alpha(-4\mathrm{i}\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}_{\dot{\alpha}}\eta^2) + (\bar{\mathbf{E}}_{\bar{\theta}})_{\dot{\alpha}}(-4\mathrm{i}\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\eta_\alpha\bar{\eta}^2) , \end{aligned} \tag{7.15b}$$

$$\begin{aligned} \mathbf{\Omega}_{\mathrm{U}(1)} &= \mathbf{E}^a(\,3\omega^2\mathrm{e}^{\omega y}\bar{\eta}_{\dot{\beta}}\tilde{\sigma}_a^{\dot{\beta}\beta}\eta_\beta) + (\mathbf{E}_\theta)^\alpha(\,3\mathrm{i}\omega\mathrm{e}^{\frac{1}{2}\omega y}\eta_\alpha) + (\bar{\mathbf{E}}_{\bar{\theta}})_{\dot{\alpha}}(-3\mathrm{i}\omega\mathrm{e}^{\frac{1}{2}\omega y}\bar{\eta}^{\dot{\alpha}}) \\ &\quad + (\mathbf{E}_\eta)^\alpha(\,6\mathrm{i}\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\eta_\alpha\bar{\eta}^2) + (\bar{\mathbf{E}}_{\bar{\eta}})_{\dot{\alpha}}(-6\mathrm{i}\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}^{\dot{\alpha}}\eta^2) . \end{aligned} \tag{7.15c}$$

7.2 SO(4,1)×U(1) covariance

To better understand the relation between the above coset construction and the $\mathrm{AdS}^{5|8}$ supergeometry of section 2, it is necessary to figure out the precise meaning of the SO(4,1)×U(1) covariance of the vielbein and the connection. We will use several results which are collected in Appendix A and concern the reduction of 5D spinors into 4D ones.

First of all, let us recall that choosing $g = h \in H$ in relations (7.10, 7.11) gives $\hat{h} = h = \text{const}$, and the group transformations (7.11) reduce to

$$\mathbf{E}' = h\,\mathbf{E}\,h^{-1}\,, \quad \mathbf{\Omega}' = h\,\mathbf{\Omega}\,h^{-1}\,, \qquad h \in \mathrm{SO}(4,1)\times\mathrm{U}(1)\,. \tag{7.16}$$

In particular, a 5D Lorentz transformation acts as follows:

$$\mathbf{E}' = \Lambda\,\mathbf{E}\,\Lambda^{-1}\,, \quad \mathbf{\Omega}' = \Lambda\,\mathbf{\Omega}\,\Lambda^{-1}\,, \tag{7.17}$$

where

$$\Lambda = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\Lambda_{\hat\alpha}{}^{\hat\beta}$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\,, \qquad \Lambda_{\hat\alpha}{}^{\hat\beta} = \Big[\exp\Big(\frac{1}{2}\Lambda^{\hat c\hat d}(\Sigma_{\hat c\hat d})\Big)\Big]_{\hat\alpha}{}^{\hat\beta}\,. \tag{7.18}$$

This transformation law allows us to combine components of the connection into five-dimensional vector and spinor. Explicitly, we can write

$$\mathbf{E} = \left(\begin{array}{c|c} -\frac{\mathrm{i}}{2}\omega\,\mathbf{E}^{\hat a}(\Gamma_{\hat a})_{\hat\alpha}{}^{\hat\beta} & 2\omega^{\frac12}\mathbf{E}_{\hat\alpha} \\ \hline 2\omega^{\frac12}\bar{\mathbf{E}}^{\hat\beta} & 0 \end{array}\right)\,, \tag{7.19}$$

$$\mathbf{\Omega} = \frac{1}{2}\mathbf{\Omega}^{\hat a\hat b}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$(\Sigma_{\hat a\hat b})_{\hat\alpha}{}^{\hat\beta}$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 0 \end{array}\right) + \mathrm{i}\,\mathbf{\Omega}_{\mathrm{U}(1)}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$-\frac13\delta_{\hat\alpha}{}^{\hat\beta}$}} & 0 \\ & & 0 \\ \hline 0 & 0 & -\frac43 \end{array}\right)\,, \tag{7.20}$$

where

$$\mathbf{E}^{\hat a} = (\mathbf{E}^a, \mathbf{E}^5) = (\mathbf{E}^a, \mathbf{E}_y)\,, \tag{7.21a}$$

$$\mathbf{E}_{\hat\alpha} = \begin{pmatrix} (\mathbf{E}_\eta)_\alpha \\ (\bar{\mathbf{E}}_{\bar\theta})^{\dot\alpha} \end{pmatrix}\,, \quad \bar{\mathbf{E}}^{\hat\alpha} = \Big((\mathbf{E}_\theta)^\alpha, (\bar{\mathbf{E}}_{\bar\eta})_{\dot\alpha}\Big)\,, \tag{7.21b}$$

$$\mathbf{\Omega}^{\hat a\hat b} = (\mathbf{\Omega}^{ab}, \mathbf{\Omega}^{a5})\,, \tag{7.21c}$$

$$\mathbf{\Omega}^{ab} = -(\sigma^{ab})_\beta{}^\alpha\mathbf{\Omega}_\alpha{}^\beta + (\tilde\sigma^{ab})^{\dot\beta}{}_{\dot\alpha}\bar{\mathbf{\Omega}}^{\dot\alpha}{}_{\dot\beta}\,, \quad \mathbf{\Omega}^{a5} = -\tfrac12(\tilde\sigma^a)^{\dot\alpha\alpha}\mathbf{\Omega}_{\alpha\dot\alpha}\,. \tag{7.21d}$$

Note that $\mathbf{E}^{\hat a}$, $\mathbf{\Omega}_{\hat a\hat b} = -\mathbf{\Omega}_{\hat b\hat a}$ and $\mathbf{\Omega}_{\mathrm{U}(1)}$ are real. It follows that $\mathbf{E}^{\hat a}$, $\mathbf{E}_{\hat\alpha}$, $\bar{\mathbf{E}}^{\hat\alpha}$, $\mathbf{\Omega}_{\hat a\hat b}$ and $\mathbf{\Omega}_{\mathrm{U}(1)}$ transform under the 5D Lorentz group $\mathrm{SO}(4,1)$ respectively as a vector, a Dirac spinor, its Dirac conjugate spinor, an antisymmetric two-tensor and a scalar. Due to (7.21a) we can identify

$$x^5 \equiv y\,. \tag{7.22}$$

Note also that we can combine the two spinors $\mathbf{E}_{\hat\alpha}$ and $\bar{\mathbf{E}}^{\hat\alpha}$ into a 5D pseudo-Majorana spinor defined as follows:

$$\mathbf{E}_i^{\hat\alpha} = (\mathbf{E}_i^{\alpha}, -\bar{\mathbf{E}}_{i\dot\alpha}) \ , \tag{7.23}$$

$$\mathbf{E}_{\underline{1}}^{\alpha} = (\mathbf{E}_\theta)^\alpha \ , \quad \mathbf{E}_{\underline{2}}^{\alpha} = (\mathbf{E}_\eta)^\alpha \ , \quad \bar{\mathbf{E}}_{\dot\alpha}^{\underline{1}} = (\bar{\mathbf{E}}_{\bar\theta})_{\dot\alpha} \ , \quad \bar{\mathbf{E}}_{\dot\alpha}^{\underline{2}} = (\bar{\mathbf{E}}_{\bar\eta})_{\dot\alpha} \ . \tag{7.24}$$

It remains to consider the transformation properties of the vielbein and the connection under the U(1) part of the isotropy group. In accordance with (7.16), they transform as

$$\begin{aligned} &\mathbf{E}' = \Sigma\, \mathbf{E}\, \Sigma^{-1} \ , \quad \mathbf{\Omega}' = \Sigma\, \mathbf{\Omega}\, \Sigma^{-1} \ , \\ &\Sigma = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\left[\exp(-\tfrac{1}{3}\phi\,\mathrm{i}\,\delta)\right]_{\hat\alpha}{}^{\hat\beta}$}} & 0 \\ & & 0 \\ \hline 0 & 0 & \mathrm{e}^{-\frac{4}{3}\phi\,\mathrm{i}} \end{array}\right) \ . \end{aligned} \tag{7.25}$$

Clearly $\mathbf{\Omega}$ is invariant under the U(1) transformation, while $\mathbf{E}$ transforms as

$$\mathbf{E}' = \left(\begin{array}{c|c} -\frac{\mathrm{i}}{2}\omega\, \mathbf{E}^{\hat a}(\Gamma_{\hat a})_{\hat\alpha}{}^{\hat\beta} & 2\omega^{\frac{1}{2}}\left(\mathrm{e}^{\phi\mathrm{i}}\, \mathbf{E}_{\hat\alpha}\right) \\ \hline 2\omega^{\frac{1}{2}}\left(\mathrm{e}^{-\phi\mathrm{i}}\, \bar{\mathbf{E}}^{\hat\beta}\right) & 0 \end{array}\right) \ , \tag{7.26}$$

and hence $\mathbf{E}^{\hat a}$ is invariant. Note also that (7.25) induces induces the following transformation of $\mathbf{E}^i_{\hat\alpha}$:

$$\mathbf{E}'^{\hat\alpha}_i = \left[\exp(-\phi \mathrm{i} J)\right]_i{}^j \mathbf{E}_j^{\hat\alpha} \ , \qquad J_i{}^j = (\sigma_3)_i{}^j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{7.27}$$

7.3 Representation of covariant derivatives

With the vielbein and the connection having been introduced, we can now construct the covariant derivatives

$$\begin{aligned} \mathcal{D}_{\hat A} &= E_{\hat A} + \mathrm{i}\,\Phi_{\hat A}\, J + \frac{1}{2}\,\Omega_{\hat A}{}^{\hat b\hat c}\, M_{\hat b\hat c} = E_{\hat A} + \mathrm{i}\,\Phi_{\hat A}\, J + \frac{1}{2}\,\Omega_{\hat A}{}^{bc}\, M_{bc} + \Omega_{\hat A}{}^{b5}\, M_{b5} \\ &= (\mathcal{D}_{\hat a}, \mathcal{D}^i_{\hat\alpha}) = (\mathcal{D}_a, \mathcal{D}_5, \mathcal{D}^{\underline{1}}_\alpha, \bar{\mathcal{D}}^{\underline{1}\dot\alpha}, \mathcal{D}^{\underline{2}}_\alpha, \bar{\mathcal{D}}^{\underline{2}\dot\alpha}) \ . \end{aligned} \tag{7.28}$$

The vector fields $E_{\hat A}$ are defined by

$$\begin{aligned} E_{\hat A} &= \left(E_{\hat a}, E^i_{\hat\alpha}\right) = E_{\hat A}{}^M D_M \ , \\ D_M &= \left(\partial_m, \frac{\partial}{\partial y}, D_\mu, \bar D^{\dot\mu}, \frac{\partial}{\partial\eta^\mu}, \frac{\partial}{\partial\bar\eta_{\dot\mu}}\right) = E_M{}^{\hat A} E_{\hat A} \ . \end{aligned} \tag{7.29}$$

Here the supermatrices $E_{\hat{A}}{}^M$ and $E_M{}^{\hat{A}}$ have been defined in subsection 7.1. It should be pointed out that $(\partial_m, D_\mu, \bar{D}^{\dot{\mu}})$ are the 4D $\mathcal{N}=1$ flat superspace covariant derivatives, $D_\mu = \frac{\partial}{\partial\theta^\mu} + \mathrm{i}\bar{\theta}^{\dot{\mu}}\partial_{\mu\dot{\mu}}$ and $\bar{D}^{\dot{\mu}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\mu}}} + \mathrm{i}\theta_\mu\tilde{\partial}^{\dot{\mu}\mu}$. Furthermore, the connection supefields in $\mathcal{D}_{\hat{A}}$ are defined as

$$\mathbf{\Omega}_{\mathrm{U}(1)} = \mathbf{E}^{\hat{A}}\,\Phi_{\hat{A}}\ , \qquad \mathbf{\Omega}^{\hat{a}\hat{b}} = \mathbf{E}^{\hat{A}}\,\Omega_{\hat{A}}{}^{\hat{a}\hat{b}}\ . \tag{7.30}$$

It can be shown that the explicit expressions for the covariant derivatives are as follows:

$$\begin{aligned}
\mathcal{D}_a &= \mathrm{e}^{\omega y}(1+\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2)\partial_a - \mathrm{i}\omega\mathrm{e}^{\omega y}(\bar{\eta}\tilde{\sigma}_a)^\mu D_\mu - \mathrm{i}\omega\mathrm{e}^{\omega y}(\eta\sigma_a)_{\dot{\mu}}\bar{D}^{\dot{\mu}} \\
&\quad + 2\mathrm{i}\omega^2\mathrm{e}^{\omega y}\eta^2(\bar{\eta}\tilde{\sigma}_a)^\mu\frac{\partial}{\partial\eta^\mu} + 2\mathrm{i}\omega^2\mathrm{e}^{\omega y}\bar{\eta}^2(\eta\sigma_a)_{\dot{\mu}}\frac{\partial}{\partial\bar{\eta}_{\dot{\mu}}} \\
&\quad - 3\mathrm{i}\omega^2\mathrm{e}^{\omega y}(\eta\sigma_a\bar{\eta})J + \omega^2\mathrm{e}^{\omega y}\eta_{ab}\varepsilon^{bcde}(\eta\sigma_c\bar{\eta})M_{de} - \omega(1+2\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2)M_{a5}\ ,
\end{aligned} \tag{7.31a}$$

$$\mathcal{D}_5 = \frac{\partial}{\partial y}\ , \tag{7.31b}$$

$$\begin{aligned}
\mathcal{D}^{\underline{1}}_\alpha &= \mathrm{e}^{\frac{1}{2}\omega y}(1-2\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2)D_\alpha - 2\omega\mathrm{e}^{\frac{1}{2}\omega y}\eta^2\frac{\partial}{\partial\eta^\alpha} - 2\omega\mathrm{e}^{\frac{1}{2}\omega y}\eta_\alpha\bar{\eta}_{\dot{\mu}}\frac{\partial}{\partial\bar{\eta}_{\dot{\mu}}} \\
&\quad + 2\eta_\alpha\mathrm{e}^{\frac{1}{2}\omega y}\frac{\partial}{\partial y} - \mathrm{i}\omega\mathrm{e}^{\frac{5}{2}\omega y}\eta^2(\sigma^m\bar{\eta})_\alpha\partial_m \\
&\quad - 3\omega\mathrm{e}^{\frac{1}{2}\omega y}\eta_\alpha\,J + 2\omega\mathrm{e}^{\frac{1}{2}\omega y}\eta^\beta(\sigma^{ab})_{\beta\alpha}M_{ab} + 2\mathrm{i}\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\eta^2(\sigma^a\bar{\eta})_\alpha M_{a5}\ ,
\end{aligned} \tag{7.31c}$$

$$\begin{aligned}
\mathcal{D}^{\underline{2}}_\alpha &= \mathrm{e}^{-\frac{1}{2}\omega y}(1-4\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2)\frac{\partial}{\partial\eta^\alpha} + 2\omega\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}^2 D_\alpha - 2\omega\mathrm{e}^{\frac{3}{2}\omega y}\eta_\alpha\bar{\eta}_{\dot{\mu}}\bar{D}^{\dot{\mu}} + \mathrm{i}\mathrm{e}^{\frac{3}{2}\omega y}(\sigma^m\bar{\eta})_\alpha\partial_m \\
&\quad - 6\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\eta_\alpha\bar{\eta}^2 J + 2\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\eta^\beta\bar{\eta}^2(\sigma^{ab})_{\beta\alpha}M_{ab} - 2\mathrm{i}\omega\mathrm{e}^{\frac{1}{2}\omega y}(\sigma^a\bar{\eta})_\alpha M_{a5}\ ,
\end{aligned} \tag{7.31d}$$

$$\begin{aligned}
\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}} &= \mathrm{e}^{\frac{1}{2}\omega y}(1-2\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2)\bar{D}^{\dot{\alpha}} - 2\omega\mathrm{e}^{\frac{1}{2}\omega y}\bar{\eta}^2\frac{\partial}{\partial\bar{\eta}_{\dot{\alpha}}} - 2\omega\mathrm{e}^{\frac{1}{2}\omega y}\bar{\eta}^{\dot{\alpha}}\eta^\mu\frac{\partial}{\partial\eta^\mu} \\
&\quad + 2\bar{\eta}^{\dot{\alpha}}\mathrm{e}^{\frac{1}{2}\omega y}\frac{\partial}{\partial y} - \mathrm{i}\omega\mathrm{e}^{\frac{5}{2}\omega y}\bar{\eta}^2(\tilde{\sigma}^m\eta)^{\dot{\alpha}}\partial_m \\
&\quad + 3\omega\mathrm{e}^{\frac{1}{2}\omega y}\bar{\eta}^{\dot{\alpha}}\,J + 2\omega\mathrm{e}^{\frac{1}{2}\omega y}\bar{\eta}_{\dot{\beta}}(\tilde{\sigma}^{ab})^{\dot{\beta}\dot{\alpha}}M_{ab} + 2\mathrm{i}\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}^2(\tilde{\sigma}^a\eta)^{\dot{\alpha}}M_{a5}\ ,
\end{aligned} \tag{7.31e}$$

$$\begin{aligned}
\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{2}} &= \mathrm{e}^{-\frac{1}{2}\omega y}(1-4\omega^2\mathrm{e}^{2\omega y}\eta^2\bar{\eta}^2)\frac{\partial}{\partial\bar{\eta}_{\dot{\alpha}}} + 2\omega\mathrm{e}^{\frac{3}{2}\omega y}\eta^2\bar{D}^{\dot{\alpha}} - 2\omega\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}^{\dot{\alpha}}\eta^\mu D_\mu + \mathrm{i}\mathrm{e}^{\frac{3}{2}y}(\tilde{\sigma}^m\eta)^{\dot{\alpha}}\partial_m \\
&\quad + 6\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}^{\dot{\alpha}}\eta^2\,J + 2\omega^2\mathrm{e}^{\frac{3}{2}\omega y}\bar{\eta}_{\dot{\beta}}\eta^2(\tilde{\sigma}^{ab})^{\dot{\beta}\dot{\alpha}}M_{ab} - 2\mathrm{i}\omega\mathrm{e}^{\frac{1}{2}\omega y}(\tilde{\sigma}^a\eta)^{\dot{\alpha}}M_{a5}\ .
\end{aligned} \tag{7.31f}$$

It is interesting to consider a flat superspace limit, $\omega\to 0$, for the covariant derivatives. In this limit, one finds

$$\mathcal{D}_{\hat{A}}\Big|_{\omega\to 0} = \mathrm{e}^{-U}D_{\hat{A}}\,\mathrm{e}^{U}\ , \qquad U = \eta\theta + \bar{\eta}\bar{\theta}\ , \tag{7.32}$$

where $D_{\hat{A}} = (\partial_{\hat{a}}, D^i_{\hat{\alpha}})$ are 5D flat global covariant derivatives,

$$D^i_{\hat{\alpha}} = \frac{\partial}{\partial\theta^{\hat{\alpha}}_i} - \mathrm{i}\,(\Gamma^{\hat{b}})_{\hat{\alpha}\hat{\beta}}\,\theta^{\hat{\beta}i}\,\partial_{\hat{b}}\ , \tag{7.33}$$

with $\theta^{\hat{\alpha}}_i = (\theta^\alpha_i, -\bar{\theta}_{\dot{\alpha}i})$ and $\theta^\alpha_i = (\theta^\alpha, \eta^\alpha)$.

7.4 Torsion and curvature

Now, we are prepared to demonstrate that the geometry described in the present section reproduces the geometry of $\mathrm{AdS}^{5|8}$ constructed in section 2.

We proceed by recalling that, in accordance with the coset construction, the torsion $\mathbf{T}$ and curvature $\mathbf{R}$ two-forms are defined as follows:

$$\mathbf{T} \,=\, \mathrm{d}\mathbf{E} \,-\, \mathbf{\Omega}\wedge\mathbf{E} \,-\, \mathbf{E}\wedge\mathbf{\Omega}\,, \qquad \mathbf{R} \,=\, \mathrm{d}\mathbf{\Omega} \,-\, \mathbf{\Omega}\wedge\mathbf{\Omega}\,. \tag{7.34}$$

Under group transformations (7.10) they transform covariantly

$$\mathbf{T}' \,=\, \hat{h}\,\mathbf{T}\,\hat{h}^{-1}\,, \qquad \mathbf{R}' \,=\, \hat{h}\,\mathbf{R}\,\hat{h}^{-1}\,. \tag{7.35}$$

Keeping in mind the definition $\mathbf{E}+\mathbf{\Omega} = G^{-1}\mathrm{d}G$, we get

$$\mathrm{d}\mathbf{E}+\mathrm{d}\mathbf{\Omega} \,=\, G^{-1}\mathrm{d}G\wedge G^{-1}\mathrm{d}G = \mathbf{E}\wedge\mathbf{E} \,+\, \mathbf{E}\wedge\mathbf{\Omega} \,+\, \mathbf{\Omega}\wedge\mathbf{E} \,+\, \mathbf{\Omega}\wedge\mathbf{\Omega}\,, \tag{7.36}$$

from which we obtain

$$\mathrm{d}\mathbf{E} \,=\, (\mathbf{E}\wedge\mathbf{E})|_{\mathcal{G}-\mathcal{H}} \,+\, \mathbf{E}\wedge\mathbf{\Omega} \,+\, \mathbf{\Omega}\wedge\mathbf{E}\,, \qquad \mathrm{d}\mathbf{\Omega} \,=\, (\mathbf{E}\wedge\mathbf{E})|_{\mathcal{H}} \,+\, \mathbf{\Omega}\wedge\mathbf{\Omega}\,, \tag{7.37}$$

since $(\mathbf{E}\wedge\mathbf{\Omega}+\mathbf{\Omega}\wedge\mathbf{E}) \in \mathcal{G}-\mathcal{H}$ and $\mathbf{\Omega}\wedge\mathbf{\Omega}\in\mathcal{H}$. Using the previous formulae we are able to see that the torsion and curvature two-forms are given by simple expressions

$$\mathbf{T} \,=\, (\mathbf{E}\wedge\mathbf{E})|_{\mathcal{G}-\mathcal{H}}\,, \qquad \mathbf{R} \,=\, (\mathbf{E}\wedge\mathbf{E})|_{\mathcal{H}}\,. \tag{7.38}$$

Therefore, it remains to compute $\mathbf{E}\wedge\mathbf{E}$.

Direct calculations give

$$\mathbf{E}\wedge\mathbf{E} \,=\, \left(\begin{array}{c|c} \frac{1}{2}\,\omega^2\,\mathbf{E}^{\hat a}\wedge\mathbf{E}^{\hat b}(\Sigma_{\hat a\hat b})_{\hat\alpha}{}^{\hat\beta} + 4\omega\,\mathbf{E}_{\underline{2}\hat\alpha}\wedge\mathbf{E}_{\underline{1}}^{\hat\beta} & -\mathrm{i}\omega^{\frac{3}{2}}\,\mathbf{E}^{\hat a}\wedge\mathbf{E}_{\underline{2}\hat\gamma}\,(\Gamma_{\hat a})_{\hat\alpha}{}^{\hat\gamma} \\ \hline -\mathrm{i}\omega^{\frac{3}{2}}\,\mathbf{E}_{\underline{1}}^{\hat\gamma}\wedge\mathbf{E}^{\hat b}\,(\Gamma_{\hat b})_{\hat\gamma}{}^{\hat\beta} & 4\omega\,\mathbf{E}_{\underline{1}}^{\hat\gamma}\wedge\mathbf{E}_{\underline{2}\hat\gamma} \end{array}\right)\,, \tag{7.39}$$

and this we should represent as $\mathbf{E}\wedge\mathbf{E} = (\mathbf{E}\wedge\mathbf{E})|_{\mathcal{G}-\mathcal{H}} + (\mathbf{E}\wedge\mathbf{E})|_{\mathcal{H}}$. We end up with

$$\mathbf{T} \,=\, \left(\begin{array}{c|c} -\frac{\mathrm{i}}{2}\omega\,\mathbf{T}^{\hat a}(\Gamma_{\hat a})_{\hat\alpha}{}^{\hat\beta} & 2\omega^{\frac{1}{2}}\,\mathbf{T}_{\underline{2}\hat\alpha} \\ \hline 2\omega^{\frac{1}{2}}\,\mathbf{T}_{\underline{1}}^{\hat\beta} & 0 \end{array}\right)\,, \tag{7.40}$$

$$\mathbf{R} \,=\, \frac{1}{2}\mathbf{R}^{\hat a\hat b}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{(\Sigma_{\hat a\hat b})_{\hat\alpha}{}^{\hat\beta}} & \begin{array}{c}0\\0\end{array} \\ \hline 0 & 0 & 0 \end{array}\right) \,+\, \mathrm{i}\,\mathbf{R}_{\mathrm{U}(1)}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{-\frac{1}{3}\delta_{\hat\alpha}{}^{\hat\beta}} & \begin{array}{c}0\\0\end{array} \\ \hline 0 & 0 & -\frac{4}{3} \end{array}\right)\,, \tag{7.41}$$

where

$$\mathbf{T}^{\hat{a}} = \frac{1}{2}\,\mathbf{E}_k^{\hat{\gamma}} \wedge \mathbf{E}_j^{\hat{\beta}} \left(2\mathrm{i}\,\varepsilon^{jk}(\Gamma^{\hat{a}})_{\hat{\beta}\hat{\gamma}}\right) , \tag{7.42}$$

$$\mathbf{T}_i^{\hat{\alpha}} = \frac{1}{2}\left[\mathbf{E}^{\hat{c}} \wedge \mathbf{E}_j^{\hat{\beta}} \left(\frac{\mathrm{i}}{2}\omega(\sigma_3)_i{}^j(\Gamma_{\hat{c}})_{\hat{\beta}}{}^{\hat{\alpha}}\right) + \mathbf{E}_k^{\hat{\gamma}} \wedge \mathbf{E}^{\hat{b}} \left(-\frac{\mathrm{i}}{2}\omega(\sigma_3)_i{}^k(\Gamma_{\hat{b}})_{\hat{\gamma}}{}^{\hat{\alpha}}\right)\right] , \tag{7.43}$$

$$\mathbf{R}^{\hat{a}\hat{b}} = \frac{1}{2}\left[\mathbf{E}^{\hat{d}} \wedge \mathbf{E}^{\hat{c}}\,\omega^2(-\delta_{\hat{c}}^{\hat{a}}\delta_{\hat{d}}^{\hat{b}} + \delta_{\hat{c}}^{\hat{b}}\delta_{\hat{d}}^{\hat{a}}) + \mathbf{E}_l^{\hat{\delta}} \wedge \mathbf{E}_k^{\hat{\gamma}} \left(-4\omega\varepsilon^{ki}(\sigma_3)_i{}^l(\Sigma^{\hat{a}\hat{b}})_{\hat{\gamma}\hat{\delta}}\right)\right] , \tag{7.44}$$

$$\mathbf{R}_{\mathrm{U}(1)} = \frac{1}{2}\,\mathbf{E}_l^{\hat{\delta}} \wedge \mathbf{E}_k^{\hat{\gamma}} \left(3\mathrm{i}\omega\varepsilon^{kl}\varepsilon_{\hat{\gamma}\hat{\delta}}\right) . \tag{7.45}$$

Using standard superform definitions [19], we define the components of the torsion and curvature as follows:

$$\mathbf{T}^{\hat{A}} = \tfrac{1}{2}\,\mathbf{E}^{\hat{C}} \wedge \mathbf{E}^{\hat{B}}\,T_{\hat{B}\hat{C}}{}^{\hat{A}} , \tag{7.46}$$

$$\mathbf{R}^{\hat{a}\hat{b}} = \tfrac{1}{2}\,\mathbf{E}^{\hat{D}} \wedge \mathbf{E}^{\hat{C}}\,R_{\hat{C}\hat{D}}{}^{\hat{a}\hat{b}} , \quad \mathbf{R}_{\mathrm{U}(1)} = \tfrac{1}{2}\,\mathbf{E}^{\hat{D}} \wedge \mathbf{E}^{\hat{C}}\,(R_{\mathrm{U}(1)})_{\hat{C}\hat{D}} . \tag{7.47}$$

Now, let us return to the covariant derivatives described in section 2. Their algebra given by eqs. (2.21a–2.21c) can be represented concisely as

$$\left[\mathcal{D}_{\hat{A}}, \mathcal{D}_{\hat{B}}\right\} = -T_{\hat{A}\hat{B}}{}^{\hat{C}}\mathcal{D}_{\hat{C}} + \mathrm{i}\,(R_{\mathrm{U}(1)})_{\hat{A}\hat{B}}\,J + \frac{1}{2}\,R_{\hat{A}\hat{B}}{}^{\hat{c}\hat{d}}M_{\hat{c}\hat{d}} . \tag{7.48}$$

Comparing (2.21a–2.21c) with eqs. (7.42–7.45), we find that all the components of the torsion and curvature coincide provided

$$J^i{}_j = (\sigma_3)^i{}_j . \tag{7.49}$$

This completes our analysis of the coset construction.

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A 5D Conventions

Our 5D notation and conventions correspond to [10]. The 5D gamma-matrices $\Gamma_{\hat{m}} = (\Gamma_m, \Gamma_5)$, with $m = 0, 1, 2, 3$, are defined by

$$\{\Gamma_{\hat{m}}\,,\,\Gamma_{\hat{n}}\} = -2\eta_{\hat{m}\hat{n}}\,\mathbb{1} , \qquad (\Gamma_{\hat{m}})^\dagger = \Gamma_0\,\Gamma_{\hat{m}}\,\Gamma_0 \tag{A.1}$$

are chosen in accordance with

$$(\Gamma_m)_{\hat\alpha}{}^{\hat\beta} = \begin{pmatrix} 0 & (\sigma_m)_{\alpha\dot\beta} \\ (\tilde\sigma_m)^{\dot\alpha\beta} & 0 \end{pmatrix} \ , \qquad (\Gamma_5)_{\hat\alpha}{}^{\hat\beta} = \begin{pmatrix} -\mathrm{i}\,\delta_\alpha{}^\beta & 0 \\ 0 & \mathrm{i}\,\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} \ , \tag{A.2}$$

such that $\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_5 = \mathbb{1}$. The charge conjugation matrix, $C = (\varepsilon^{\hat\alpha\hat\beta})$, and its inverse, $C^{-1} = C^\dagger = (\varepsilon_{\hat\alpha\hat\beta})$ are defined by

$$C\,\Gamma_{\hat m}\,C^{-1} = (\Gamma_{\hat m})^{\mathrm{T}} \ , \qquad \varepsilon^{\hat\alpha\hat\beta} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot\alpha\dot\beta} \end{pmatrix} \ , \quad \varepsilon_{\hat\alpha\hat\beta} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & -\varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} \ . \tag{A.3}$$

The antisymmetric matrices $\varepsilon^{\hat\alpha\hat\beta}$ and $\varepsilon_{\hat\alpha\hat\beta}$ are used to raise and lower the four-component spinor indices.

A Dirac spinor, $\Psi = (\Psi_{\hat\alpha})$, and its Dirac conjugate, $\bar\Psi = (\bar\Psi^{\hat\alpha}) = \Psi^\dagger\,\Gamma_0$, look like

$$\Psi_{\hat\alpha} = \begin{pmatrix} \psi_\alpha \\ \bar\phi^{\dot\alpha} \end{pmatrix} \ , \qquad \bar\Psi^{\hat\alpha} = (\phi^\alpha\ ,\ \bar\psi_{\dot\alpha}) \ . \tag{A.4}$$

One can now combine $\bar\Psi^{\hat\alpha} = (\phi^\alpha, \bar\psi_{\dot\alpha})$ and $\Psi^{\hat\alpha} = \varepsilon^{\hat\alpha\hat\beta}\Psi_{\hat\beta} = (\psi^\alpha, -\bar\phi_{\dot\alpha})$ into a SU(2) doublet,

$$\Psi_i^{\hat\alpha} = (\Psi_i^\alpha, -\bar\Psi_{\dot\alpha i}) \ , \qquad (\Psi_i^\alpha)^* = \bar\Psi^{\dot\alpha i} \ , \qquad i = \underline{1}, \underline{2} \ , \tag{A.5}$$

with $\Psi_{\underline{1}}^\alpha = \phi^\alpha$ and $\Psi_{\underline{2}}^\alpha = \psi^\alpha$. It is understood that the SU(2) indices are raised and lowered by ε^{ij} and ε_{ij}, $\varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1$, in the standard fashion: $\Psi^{\hat\alpha i} = \varepsilon^{ij}\Psi_j^{\hat\alpha}$. The Dirac spinor $\Psi^i = (\Psi^i_{\hat\alpha})$ satisfies the pseudo-Majorana condition $\bar\Psi_i{}^{\mathrm{T}} = C\,\Psi_i$. This will be concisely represented as

$$(\Psi^i_{\hat\alpha})^* = \Psi_i^{\hat\alpha} \ . \tag{A.6}$$

With the definition $\Sigma_{\hat m\hat n} = -\Sigma_{\hat n\hat m} = -\frac{1}{4}[\Gamma_{\hat m}, \Gamma_{\hat n}]$, the matrices $\{\mathbb{1}, \Gamma_{\hat m}, \Sigma_{\hat m\hat n}\}$ form a basis in the space of 4×4 matrices. The matrices $\varepsilon_{\hat\alpha\hat\beta}$ and $(\Gamma_{\hat m})_{\hat\alpha\hat\beta}$ are antisymmetric, $\varepsilon^{\hat\alpha\hat\beta}\,(\Gamma_{\hat m})_{\hat\alpha\hat\beta} = 0$, while the matrices $(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}$ are symmetric.

It is useful to write explicitly the 4D reduction of these matrices

$$(\Gamma_m)_{\hat\alpha\hat\beta} = \begin{pmatrix} 0 & -(\sigma_m)_\alpha{}^{\dot\beta} \\ (\sigma_m)_\beta{}^{\dot\alpha} & 0 \end{pmatrix} \ , \quad (\Gamma_5)_{\hat\alpha\hat\beta} = \begin{pmatrix} \mathrm{i}\,\varepsilon_{\alpha\beta} & 0 \\ 0 & \mathrm{i}\,\varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} \ , \tag{A.7}$$

$$(\Sigma_{mn})_{\hat\alpha}{}^{\hat\beta} = \begin{pmatrix} (\sigma_{mn})_\alpha{}^\beta & 0 \\ 0 & (\tilde\sigma_{mn})^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} \ , \quad (\Sigma_{m5})_{\hat\alpha}{}^{\hat\beta} = \begin{pmatrix} 0 & -\frac{\mathrm{i}}{2}(\sigma_m)_{\alpha\dot\beta} \\ \frac{\mathrm{i}}{2}(\tilde\sigma_m)^{\dot\alpha\beta} & 0 \end{pmatrix} \ , \tag{A.8}$$

$$(\Sigma_{mn})_{\hat\alpha\hat\beta} = \begin{pmatrix} (\sigma_{mn})_{\alpha\beta} & 0 \\ 0 & -(\tilde\sigma_{mn})^{\dot\alpha\dot\beta} \end{pmatrix} \ , \quad (\Sigma_{m5})_{\hat\alpha\hat\beta} = \begin{pmatrix} 0 & \frac{\mathrm{i}}{2}(\sigma_m)_\alpha{}^{\dot\beta} \\ \frac{\mathrm{i}}{2}(\sigma_m)_\beta{}^{\dot\alpha} & 0 \end{pmatrix} \ , \tag{A.9}$$

where $(\sigma_{mn})_\alpha{}^\beta = -\frac{1}{4}(\sigma_m\tilde\sigma_n - \sigma_n\tilde\sigma_m)_\alpha{}^\beta$ and $(\tilde\sigma_{mn})^{\dot\alpha}{}_{\dot\beta} = -\frac{1}{4}(\tilde\sigma_m\sigma_n - \tilde\sigma_n\sigma_m)^{\dot\alpha}{}_{\dot\beta}$.

Given a 5-vector $V^{\hat m}$ and an antisymmetric tensor $F^{\hat m\hat n} = -F^{\hat n\hat m}$, we can equivalently represent them as the bi-spinors $V = V^{\hat m}\,\Gamma_{\hat m}$ and $F = \frac{1}{2}F^{\hat m\hat n}\,\Sigma_{\hat m\hat n}$ with the following symmetry properties

$$V_{\hat\alpha\hat\beta} = -V_{\hat\beta\hat\alpha}\ , \quad \varepsilon^{\hat\alpha\hat\beta}\,V_{\hat\alpha\hat\beta} = 0\ , \qquad F_{\hat\alpha\hat\beta} = F_{\hat\beta\hat\alpha}\ . \tag{A.10}$$

The two equivalent descriptions $V_{\hat m} \leftrightarrow V_{\hat\alpha\hat\beta}$ and and $F_{\hat m\hat n} \leftrightarrow F_{\hat\alpha\hat\beta}$ are explicitly described as follows:

$$\begin{aligned} V_{\hat\alpha\hat\beta} &= V^{\hat m}\,(\Gamma_{\hat m})_{\hat\alpha\hat\beta}\ , \qquad & V_{\hat m} &= -\frac{1}{4}\,(\Gamma_{\hat m})^{\hat\alpha\hat\beta}\,V_{\hat\alpha\hat\beta}\ , \\ F_{\hat\alpha\hat\beta} &= \frac{1}{2}F^{\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\ , \qquad & F_{\hat m\hat n} &= (\Sigma_{\hat m\hat n})^{\hat\alpha\hat\beta}\,F_{\hat\alpha\hat\beta}\ . \end{aligned} \tag{A.11}$$

These results can be easily checked using the identities

$$\begin{aligned} \varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta} &= \varepsilon_{\hat\alpha\hat\beta}\,\varepsilon_{\hat\gamma\hat\delta} + \varepsilon_{\hat\alpha\hat\gamma}\,\varepsilon_{\hat\delta\hat\beta} + \varepsilon_{\hat\alpha\hat\delta}\,\varepsilon_{\hat\beta\hat\gamma}\ , \\ \varepsilon_{\hat\alpha\hat\gamma}\,\varepsilon_{\hat\beta\hat\delta} - \varepsilon_{\hat\alpha\hat\delta}\,\varepsilon_{\hat\beta\hat\gamma} &= -\frac{1}{2}\,(\Gamma^{\hat m})_{\hat\alpha\hat\beta}\,(\Gamma_{\hat m})_{\hat\gamma\hat\delta} + \frac{1}{2}\,\varepsilon_{\hat\alpha\hat\beta}\,\varepsilon_{\hat\gamma\hat\delta}\ , \end{aligned} \tag{A.12}$$

and therefore

$$\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta} = \frac{1}{2}\,(\Gamma^{\hat m})_{\hat\alpha\hat\beta}\,(\Gamma_{\hat m})_{\hat\gamma\hat\delta} + \frac{1}{2}\,\varepsilon_{\hat\alpha\hat\beta}\,\varepsilon_{\hat\gamma\hat\delta}\ , \tag{A.13}$$

with $\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ the completely antisymmetric fourth-rank tensor.

Complex conjugation gives

$$(\varepsilon_{\hat\alpha\hat\beta})^* = -\varepsilon^{\hat\alpha\hat\beta}\ , \qquad (V_{\hat\alpha\hat\beta})^* = V^{\hat\alpha\hat\beta}\ , \qquad (F_{\hat\alpha\hat\beta})^* = F^{\hat\alpha\hat\beta}\ , \tag{A.14}$$

provided $V^{\hat m}$ and $F^{\hat m\hat n}$ are real.

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